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Jacobi approximations in non-uniformly Jacobi-weighted Sobolev spaces

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Abstract

Jacobi approximations in non-uniformly Jacobi-weighted Sobolev spaces are investigated. Some results on orthogonal projections and interpolations are established. Explicit expressions describing the dependence of approximation results on the parameters of Jacobi polynomials are given. These results serve as an important tool in the analysis of numerous quadratures and numerical methods for differential and integral equations.

MSC: 41A10; 41A25

Keywords: Jacobi approximations; Orthogonal projections; Interpolations

1. Introduction

The Jacobi polynomials $J_l^{(\alpha,\beta)}(x)$ play important roles in mathematical analysis and its applications, see [1,27,28]. In the early work, one only considered Jacobi

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approximations in uniformly weighted Sobolev spaces. In other words, the weight is uniform for all derivatives involved in their norms. This fact limits their applications. For instance, we consider the following equation in cylindrical coordinates,

$$\frac{1}{r}\partial_r(r\partial_r v) + \frac{1}{r^2}\partial_\theta^2 v + \partial_z^2 v + \mu v = f.$$

In its weak formulation, the weights for $\partial_r v, \partial_\theta v, \partial_z v$ and v are $r, \frac{1}{r}, r$ and r, respectively, see [5]. So we cannot use Jacobi approximations in uniformly weighted spaces to deal with this problem properly. It is also difficult to use such approximations for singular differential equations, see [24].

In the past decade, Jacobi approximations developed again because of several reasons. Firstly, Gegenbauer approximations were successfully used for removing Gibbs phenomenon, see [12]. Next, the usual Gauss-type interpolations are not applicable to quadratures involving derivatives of functions at endpoints, and so we need to study certain Jacobi interpolations, see [10]. Thirdly, in the numerical analysis of finite element methods, one used some results on Jacobi approximations, see [2,22,23,26]. In particular, the Legendre and Chebyshev approximations have been widely used for spectral methods of non-singular differential equations, see [6,7,11,13]. Recently, some authors applied Jacobi approximations directly to singular problems and differential equations on unbounded domains and axisymmetric domains, see [5,14–17,20]. Furthermore, Dubiner [9] investigated an orthogonal approximation on a triangle in which the base functions are the products of two Jacobi polynomials, also see [23]. Jacobi approximations were also used for the numerical analysis of some rational approximations, see [18,19].

As we know, the more precise the results on Jacobi approximations, the more accurate the error estimates of related numerical algorithms. Canuto and Quarteroni [8] first studied the Legendre and Chebyshev approximations in Sobolev spaces. Bernardi and Maday [6] developed symmetric Jacobi approximations ($\alpha = \beta$) in uniformly weighted Sobolev spaces. However in many practical problems, the coefficients of derivatives of unknown functions involved in differential equations degenerate in different ways. Therefore we need to study various orthogonal projections in non-uniformly Jacobi-weighted Sobolev spaces, in which the weights for different derivatives appearing in the expressions of norms are different. Babuška and Guo [3], Guo [16,17], and Guo and Wang [20] developed such approximations. But the results in [3] are valid only for symmetric Jacobi approximations in the standard Jacobi-weighted Sobolev spaces in which the weight for derivative of order k is the product of the weight for function itself and $(1-x^2)^k$. This is not the most appropriate in some applications. On the other hand, the results of Guo [16,17] and Guo and Wang [20] do not seem optimal. Furthermore, the existing results are of the form,

$$||Q_N v - v||_{B_1} \leqslant c_* N^{-\lambda} ||v||_{B_2}, \quad \lambda \geqslant 0, \tag{1.1}$$

where B_1 is a certain Sobolev space, B_2 is a related Sobolev or Besov space, Q_N is an orthogonal projection or interpolation upon the set of polynomials of degree at most N. The generic positive constant c_* does not depend on N and v, but depends on α

and β implicitly. Such an estimate is useful for many problems. But it is not enough sometimes. For instance, for the orthogonal approximation on a triangle, we take the base functions as the products of two Jacobi polynomials, where one of parameter of the first Jacobi polynomial is just the degree of the second one. In this case, we have to explore explicit dependence of c_* on the parameters α and β.

This paper is devoted to Jacobi approximations in non-uniformly Jacobi-weighted Sobolev spaces. We shall derive approximation results as

$$||Q_N v - v||_{B_1} \leqslant d_{N,\alpha,\beta}|v|_{B_2}, \tag{1.2}$$

where B_1 and B_2 are non-uniformly Jacobi-weighted Sobolev spaces, $d_{N,\alpha,\beta}$ is an explicit function of N, α and β, independent of v. The main advantages of this work are as follows. Firstly, the results are valid for general Jacobi approximations, and so could be applied to numerous problems. Next, all estimates are as sharp as possible. In particular, the space B_2 in (1.2) is much more reasonable than those in existing literatures, and seems optimal. This fact simplifies theoretical analysis, and leads to more precise results on various numerical methods. Finally, the explicit expressions describing the dependence of $||Q_N v - v||_{B_1}$ on α and β are presented, which open a new goal for applications of Jacobi approximations.

This paper is organized as follows. In the next section, we establish some basic results on Jacobi approximations. In Section 3, we deal with several orthogonal approximations in non-uniformly Jacobi-weighted Sobolev spaces, which are related to numerical solutions of various differential equations. In Section 4, we study Jacobi–Gauss-type interpolations which are often preferable in the numerical solutions of differential and integral equations. The final section is for some concluding remarks.

2. Preliminaries

Let $\Lambda = \{x \mid |x| < 1\}$ and $\chi(x)$ be a certain weight function. Denote by $\mathbb{N}$ the set of all non-negative integers. For any $r \in \mathbb{N}$, we define the weighted Sobolev space $H^r_\chi(\Lambda)$ in the usual way, and denote its inner product, semi-norm and norm by $(u,v)_{r,\chi}$, $|v|_{r,\chi}$ and $||v||_{r,\chi}$, respectively. In particular, $L^2_\chi(\Lambda) = H^0_\chi(\Lambda)$, $(u,v)_\chi = (u,v)_{0,\chi}$ and $||v||_\chi = ||v||_{0,\chi}$. For any real number $r = [r] + \theta, 0 < \theta < 1$, we define the interpolation space $H^r_\chi(\Lambda) = [H^{[r]+1}_\chi(\Lambda), H^{[r]}_\chi(\Lambda)]_{1-\theta}$ as in [4]. Moreover, the following Gagliardo–Nirenberg-type inequality holds (see [4] and (1.10) of [6]),

$$||v||_{r,\chi} \leqslant ||v||^{\theta}_{[r]+1,\chi} ||v||^{1-\theta}_{[r],\chi} \quad \forall v \in H^{[r]+1}_\chi(\Lambda).$$

Furthermore, the space $H^r_{0,\chi}(\Lambda)$ stands for the closure in $H^r_\chi(\Lambda)$ of the set $\mathcal{D}(\Lambda)$ consisting of all infinitely differentiable functions with compact support in Λ. When $\chi(x) \equiv 1$, we omit the subscript χ in notations as usual.

The Jacobi polynomials $J_l^{(\alpha,\beta)}(x)$, $l = 0, 1, 2, \ldots$, are the eigenfunctions of Sturm–Liouville problem

$$\partial_x((1-x)^{\alpha+1}(1+x)^{\beta+1}\partial_x v(x)) + \lambda(1-x)^{\alpha}(1+x)^{\beta}v(x) = 0, \quad x \in \Lambda, \tag{2.1}$$

with the corresponding eigenvalues $\lambda_l^{(\alpha,\beta)} = l(l+\alpha+\beta+1), l = 0, 1, 2, \ldots$. It is noted that

$$J_l^{(\alpha,\beta)}(-x) = (-1)^l J_l^{(\beta,\alpha)}(x), \quad J_l^{(\alpha,\beta)}(1) = \frac{\Gamma(l+\alpha+1)}{l!\Gamma(\alpha+1)}. \tag{2.2}$$

The Jacobi polynomials fulfill the recurrence relation

$$\partial_x J_l^{(\alpha,\beta)}(x) = \tfrac{1}{2}(l+\alpha+\beta+1)J_{l-1}^{(\alpha+1,\beta+1)}(x), \quad l \geqslant 1. \tag{2.3}$$

Let $\chi^{(\alpha,\beta)}(x) = (1-x)^{\alpha}(1+x)^{\beta}$, $\alpha, \beta > -1$. We have

$$\int_\Lambda J_l^{(\alpha,\beta)}(x) J_{l'}^{(\alpha,\beta)}(x)\chi^{(\alpha,\beta)}(x)\, dx = \gamma_l^{(\alpha,\beta)}\delta_{l,l'}, \tag{2.4}$$

where $\delta_{l,l'}$ is the Kronecker function, and

$$\gamma_l^{(\alpha,\beta)} = \frac{2^{\alpha+\beta+1}\Gamma(l+\alpha+1)\Gamma(l+\beta+1)}{(2l+\alpha+\beta+1)\Gamma(l+1)\Gamma(l+\alpha+\beta+1)}. \tag{2.5}$$

For any $v \in L^2_{\chi^{(\alpha,\beta)}}(\Lambda)$, we have that

$$v(x) = \sum_{l=0}^{\infty} \hat{v}_l^{(\alpha,\beta)} J_l^{(\alpha,\beta)}(x), \quad \hat{v}_l^{(\alpha,\beta)} = \frac{1}{\gamma_l^{(\alpha,p)}} \int_\Lambda v(x) J_l^{(\alpha,\beta)}(x)\chi^{(\alpha,\beta)}(x)\, dx.$$

Let $N \in \mathbb{N}$. We denote by $\mathcal{P}_N$ the set of all algebraic polynomials of degree at most N. Moreover, ${}_0\mathcal{P}_N = \{v \mid v \in \mathcal{P}_N, v(-1) = 0\}$ and $\mathcal{P}_N^0 = \{v \mid v \in \mathcal{P}_N, v(\pm 1) = 0\}$.

We first consider the orthogonal projection $P_{N,\alpha,\beta} : L^2_{\chi^{(\alpha,\beta)}}(\Lambda) \to \mathcal{P}_N$. It is defined by

$$(P_{N,\alpha,\beta}v - v, \phi)_{\chi^{(\alpha,\beta)}} = 0 \quad \forall \phi \in \mathcal{P}_N. \tag{2.6}$$

To derive approximation results, we introduce the Jacobi-weighted space

$$H^r_{\chi^{(\alpha,\beta)},A}(\Lambda) = \{v \mid v \text{ is measurable and } ||v||_{r,\chi^{(\alpha,\beta)},A} < \infty\}, \quad r \in \mathbb{N},$$

equipped with the following norm and semi-norm,

$$||v||_{r,\chi^{(\alpha,\beta)},A} = \left(\sum_{k=0}^{r} ||\partial_x^k v||^2_{\chi^{(\alpha+k,\beta+k)}}\right)^{1/2}, \quad |v|_{r,\chi^{(\alpha,\beta)},A} = ||\partial_x^r v||_{\chi^{(\alpha+r,\beta+r)}}.$$

Next, let $r = [r] + \theta > 0$. Since $H^{[r]}_{\chi^{(\alpha,\beta)},A}(\Lambda)$ and $H^{[r]+1}_{\chi^{(\alpha,\beta)},A}(\Lambda)$ are separable Hilbert spaces such that $H^{[r]+1}_{\chi^{(\alpha,\beta)},A}(\Lambda)$ is continuously imbedded and dense in $H^{[r]}_{\chi^{(\alpha,\beta)},A}(\Lambda)$, we can use complex interpolation as in [4] to define the space $H^r_{\chi^{(\alpha,\beta)},A}(\Lambda) =$

$[H^{[r]+1}_{\chi^{(\alpha,\beta)},A}(\Lambda), H^{[r]}_{\chi^{(\alpha,\beta)},A}(\Lambda)]_{1-\theta}$. Moreover,

$$||v||_{r,\chi^{(\alpha,\beta)},A} \leqslant ||v||^{\theta}_{[r]+1,\chi^{(\alpha,\beta)},A} ||v||^{1-\theta}_{[r],\chi^{(\alpha,\beta)},A} \quad \forall v \in H^{[r]+1}_{\chi^{(\alpha,\beta)},A}(\Lambda). \tag{2.7}$$

Theorem 2.1. *For any* $v \in H^r_{\chi^{(\alpha,\beta)},A}(\Lambda), r \in \mathbb{N}$ *and* $0 \leqslant \mu \leqslant r$,

$$||P_{N,\alpha,\beta}v - v||_{\mu,\chi^{(\alpha,\beta)},A} \leqslant c(N(N+\alpha+\beta))^{\frac{\mu-r}{2}} |v|_{r,\chi^{(\alpha,\beta)},A}. \tag{2.8}$$

Hereafter c denotes a generic positive constant independent of any function, N, α *and* β.

Proof. We have from (2.3) that for $k \in \mathbb{N}$,

$$\begin{aligned}\partial_x^k(P_{N,\alpha,\beta}v(x) - v(x)) &= -\sum_{l=N+1}^{\infty} \hat{v}_l^{(\alpha,\beta)} \partial_x^k J_l^{(\alpha,\beta)}(x)\\ &= -\sum_{l=N+1}^{\infty} c_{l,\alpha,\beta,k} \hat{v}_l^{(\alpha,\beta)} J_{l-k}^{(\alpha+k,\beta+k)}(x),\end{aligned}$$

where

$$c_{l,\alpha,\beta,k} = \frac{\Gamma(l+\alpha+\beta+k+1)}{2^k \Gamma(\alpha+\beta+1)}.$$

Thus by (2.4),

$$||\partial_x^k(P_{N,\alpha,\beta}v - v)||^2_{\chi^{(\alpha+k,\beta+k)}} = \sum_{l=N+1}^{\infty} c^2_{l,\alpha,\beta,k} (\hat{v}_l^{(\alpha,\beta)})^2 \gamma_{l-k}^{(\alpha+k,\beta+k)}. \tag{2.9}$$

Similarly,

$$||\partial_x^r v||^2_{\chi^{(\alpha+r,\beta+r)}} = \sum_{l=r+1}^{\infty} c^2_{l,\alpha,\beta,r} (\hat{v}_l^{(\alpha,\beta)})^2 \gamma_{l-r}^{(\alpha+r,\beta+r)}. \tag{2.10}$$

Using (2.5) and the Stirling formula

$$\Gamma(s+1) = \sqrt{2\pi s} s^s e^{-s}(1 + O(s^{-\frac{1}{5}})), \tag{2.11}$$

we deduce that

$$\frac{c^2_{l,\alpha,\beta,k} \gamma_{l-k}^{(\alpha+k,\beta+k)}}{c^2_{l,\alpha,\beta,r} \gamma_{l-r}^{(\alpha+r,\beta+r)}} = \frac{\Gamma(l-r+1)\Gamma(l+\alpha+\beta+k+1)}{\Gamma(l-k+1)\Gamma(l+\alpha+\beta+r+1)} \leqslant c l^{k-r} (l+\alpha+\beta)^{k-r}.$$

The above with (2.9) and (2.10) leads to that for all $k \leqslant r$,

$$|P_{N,\alpha,\beta}v - v|_{k,\chi^{(\alpha,\beta)},A} \leqslant c(N(N+\alpha+\beta))^{\frac{k-r}{2}} |v|_{r,\chi^{(\alpha,\beta)},A}.$$

This implies the desired result for $\mu \in \mathbb{N}$. For $\mu = [\mu] + \theta, 0 < \theta < 1$, we use (2.7) to verify that

$$\begin{aligned}||P_{N,\alpha,\beta}v - v||_{\mu,\chi^{(\alpha,\beta)},\Lambda} &\leqslant ||P_{N,\alpha,\beta}v - v||^{\theta}_{[\mu]+1,\chi^{(\alpha,\beta)},\Lambda} ||P_{N,\alpha,\beta}v - v||^{1-\theta}_{[\mu],\chi^{(\alpha,\beta)},\Lambda} \\ &\leqslant c(N(N+\alpha+\beta))^{\frac{\mu-r}{2}} |v|_{r,\chi^{(\alpha,\beta)},\Lambda}. \qquad \square\end{aligned}$$

We may consider Jacobi approximations for functions belonging to Jacobi-weighted Besov spaces, see [3,25], and follow the same line as in [3] to derive the corresponding result. This generalizes Theorem 2.3 of [3], since μ and r are real numbers and α could be different from β. But in this case, the norm at the right side of (2.8) becomes the norm of v, in a Jacobi-weighted Besov space. However, in duality arguments used in Section 3, we need to use the result (2.8) in which only the semi-norm $|v|_{r,\chi^{(\alpha,\beta)},\Lambda}$ appears.

An interesting application of Theorem 2.1 is stated below, which will be used in the next section. For $-1 < \alpha, \beta < 1$, let

$$\mathcal{U}_{N,\alpha,\beta}(\Lambda) = \{v \mid v = \chi^{(\alpha,\beta)}\phi, \phi \in \mathcal{P}_{N-1}\}.$$

The orthogonal projection $\mathcal{T}_{N,\alpha,\beta} : L^2_{\chi^{(-\alpha,-\beta)}}(\Lambda) \to \mathcal{U}_{N,\alpha,\beta}(\Lambda)$ is defined by

$$(\mathcal{T}_{N,\alpha,\beta}v - v, \phi)_{\chi^{(-\alpha,-\beta)}} = 0 \quad \forall \phi \in \mathcal{U}_{N,\alpha,\beta}(\Lambda). \tag{2.12}$$

In order to estimate $||\mathcal{T}_{N,\alpha,\beta}v - v||_{\chi^{(-\alpha,-\beta)}}$, we need several preparations. Firstly, we have that for any $v \in H^1_{\chi^{(-\alpha,-\beta)}}(\Lambda)$ and $-1 < \alpha, \beta < 1$,

$$\max_{x \in \bar{\Lambda}} |v(x)| \leqslant \tfrac{1}{2}(5\gamma_0^{(\alpha,\beta)})^{\frac{1}{2}} ||v||_{1,\chi^{(-\alpha,-\beta)}}. \tag{2.13}$$

Indeed, $H^1_{\chi^{(-\alpha,-\beta)}}(\Lambda) \subset C(\bar{\Lambda})$. Let $|v(x_*)| = \min_{x \in \bar{\Lambda}} |v(x)|$. Then

$$|v(x)| - |v(x_*)| \leqslant (\gamma_0^{(\alpha,\beta)})^{\frac{1}{2}} |v|_{1,\chi^{(-\alpha,-\beta)}}.$$

Moreover,

$$|v(x_*)| \leqslant \frac{1}{2}\int_{\Lambda} |v(x)|\, dx \leqslant \frac{1}{2}(\gamma_0^{(\alpha,\beta)})^{\frac{1}{2}} ||v||_{\chi^{(-\alpha,-\beta)}},$$

which leads to (2.13).

Next, let $\Lambda_1 = (0,1)$, $\Lambda_2 = (-1,0]$ and

$$\xi_a = \max(2^a, 1), \quad \zeta_{a,b} = \max(\xi_a^{\frac{1}{2}}(b+1)^{-\frac{1}{2}}, \xi_b^{\frac{1}{2}}(a+1)^{-\frac{1}{2}}), \quad a, b > -1.$$

By the Hardy inequality, for any measurable function $\psi(x)$, $a \leqslant b$ and $q < 1$,

$$\int_a^b \left(\frac{1}{b-x}\int_x^b \psi(y)\, dy\right)^2 (b-x)^q\, dx \leqslant \frac{4}{1-q}\int_a^b \psi^2(x)(b-x)^q\, dx. \tag{2.14}$$

Let $v \in H^1_{0,\chi^{(\alpha,\beta)}}(\Lambda)$. Taking $a=0, b=1$ and $q=\alpha<1$ in (2.14), we obtain that for $\beta \leqslant 2$,

$$\int_{\Lambda_1} v^2(x)\chi^{(\alpha-2,\beta-2)}(x)\,dx \leqslant \int_{\Lambda_1} v^2(x)(1-x)^{\alpha-2}\,dx \leqslant \frac{4\xi_{-\beta}}{1-\alpha}\int_{\Lambda_1}(\partial_x v(x))^2\chi^{(\alpha,\beta)}(x)\,dx.$$

A similar result holds on Λ_2. Therefore, for any $v \in H^1_{0,\chi^{(\alpha,\beta)}}(\Lambda)$ and $-1<\alpha,\beta<1$,

$$||v||_{\chi^{(\alpha,\beta)}} \leqslant ||v||_{\chi^{(\alpha-2,\beta-2)}} \leqslant 2\zeta_{-\alpha,-\beta}|v|_{1,\chi^{(\alpha,\beta)}}. \tag{2.15}$$

Lemma 2.1. *If $-1<\alpha,\beta<1$, then for any $v \in H^1_{\chi^{(-\alpha,-\beta)}}(\Lambda)$,*

$$||\mathcal{T}_{N,\alpha,\beta}v - v||_{\chi^{(-\alpha,-\beta)}} \leqslant c q_{\alpha,\beta}(N(N+\alpha+\beta))^{\frac{\max(\alpha,\beta,0)-1}{2}}||v||_{1,\chi^{(-\alpha,-\beta)}}$$

where $q_{\alpha,\beta} = \max(q^{(1)}_{\alpha,\beta}, q^{(2)}_{\alpha,\beta})$, and

$$q^{(1)}_{\alpha,\beta} = (\tfrac{1}{2}(\gamma_0^{(\alpha,\beta)}\gamma_0^{(-\alpha,-\beta)})^{\frac{1}{2}}+1)(4\zeta_{\alpha,\beta}(\alpha^2+\beta^2)^{\frac{1}{2}}+1),$$
$$q^{(2)}_{\alpha,\beta} = (\gamma_0^{(\alpha,\beta)})^{\frac{1}{2}}\max(\Gamma^{-1}(\alpha+1),\Gamma^{-1}(\beta+1)). \tag{2.16}$$

If, in addition, $v \in H^1_{0,\chi^{(-\alpha,-\beta)}}(\Lambda)$, then

$$||\mathcal{T}_{N,\alpha,\beta}v - v||_{\chi^{(-\alpha,-\beta)}} \leqslant c q^{(1)}_{\alpha,\beta}(N(N+\alpha+\beta))^{-\frac{1}{2}}|v|_{1,\chi^{(-\alpha,-\beta)}}. \tag{2.17}$$

Proof. Let

$$v^*(x) = \tfrac{1}{2}v(1)(1+x) + \tfrac{1}{2}v(-1)(1-x), \quad v^0(x) = v(x) - v^*(x), \quad x \in \bar{\Lambda}. \tag{2.18}$$

Clearly $v^0 \in H^1_{0,\chi^{(-\alpha,-\beta)}}(\Lambda)$. Moreover,

$$|v^0|_{1,\chi^{(-\alpha,-\beta)}} \leqslant |v|_{1,\chi^{(-\alpha,-\beta)}} + \tfrac{1}{2}(\gamma_0^{(-\alpha,-\beta)})^{\frac{1}{2}}\int_\Lambda |\partial_x v(x)|\,dx \leqslant (\tfrac{1}{2}(\gamma_0^{(\alpha,\beta)}\gamma_0^{(-\alpha,-\beta)})^{\frac{1}{2}}+1)|v|_{1,\chi^{(-\alpha-\beta)}}. \tag{2.19}$$

Further, let

$$v^0_N(x) = \chi^{(\alpha,\beta)}(x)P_{N-1,\alpha,\beta}(v^0(x)\chi^{(-\alpha,-\beta)}(x)).$$

Obviously $v_N^0 \in \mathcal{U}_{N,\alpha,\beta}(\Lambda)$. By Theorem 2.1 with $r=1$ and $\mu=0$, (2.15) and (2.19),

$$\begin{aligned}||v_N^0 - v^0||_{\chi^{(-\alpha,-\beta)}} &= ||P_{N-1,\alpha,\beta}(v^0\chi^{(-\alpha,-\beta)}) - v^0\chi^{(-\alpha,-\beta)}||_{\chi^{(\alpha,\beta)}}\\ &\leqslant c(N(N+\alpha+\beta))^{-\frac{1}{2}}|v^0\chi^{(-\alpha,-\beta)}|_{1,\chi^{(\alpha+1,\beta+1)}}\\ &\leqslant c(N(N+\alpha+\beta))^{-\frac{1}{2}}(|v^0|_{1,\chi^{(-\alpha,-\beta)}} + 2(\alpha^2+\beta^2)^{\frac{1}{2}}|v^0|_{\chi^{(-\alpha-2,-\beta-2)}})\\ &\leqslant c(4\zeta_{\alpha,\beta}(\alpha^2+\beta^2)^{\frac{1}{2}}+1)(N(N+\alpha+\beta))^{-\frac{1}{2}}|v^0|_{1,\chi^{(-\alpha,-\beta)}}\\ &\leqslant cq_{\alpha,\beta}^{(1)}(N(N+\alpha+\beta))^{-\frac{1}{2}}|v|_{1,\chi^{(-\alpha,-\beta)}}. \end{aligned} \tag{2.20}$$

Next, we consider the upper-bound of $||\mathcal{T}_{N,\alpha,\beta}v^* - v^*||_{\chi^{(-\alpha,-\beta)}}$. It suffices to estimate $||\mathcal{T}_{N,\alpha,\beta}w - w||_{\chi^{(-\alpha,-\beta)}},\ w = 1, x$. Due to (2.12), it can be checked that

$$\mathcal{T}_{N,\alpha,\beta}1 - 1 = -\sum_{l=N}^{\infty} \hat{d}_l \chi^{(\alpha,\beta)}(x) J_l^{(\alpha,\beta)}(x).$$

Multiplying the above by $J_l^{(\alpha,\beta)}(x),\ l \geqslant N$ and integrating the result, we obtain

$$\hat{d}_l = (\gamma_l^{(\alpha,\beta)})^{-1}\int_\Lambda J_l^{(\alpha,\beta)}(x)\,dx.$$

According to (2.7) of [20],

$$\begin{aligned}\int_\Lambda J_l^{(\alpha,\beta)}(y)\,dy &= A_l(J_{l+1}^{(\alpha,\beta)}(1) - J_{l+1}^{(\alpha,\beta)}(-1)) + B_l(J_l^{(\alpha,\beta)}(1) - J_l^{(\alpha,\beta)}(-1))\\ &\quad +\ C_l(J_{l-1}^{(\alpha,\beta)}(1) - J_{l-1}^{(\alpha,\beta)}(-1)),\end{aligned}$$

where

$$A_l = \frac{2(l+\alpha+\beta+1)}{(2l+\alpha+\beta+1)(2l+\alpha+\beta+2)}, \quad B_l = \frac{2(\alpha-\beta)}{(2l+\alpha+\beta)(2l+\alpha+\beta+2)},$$

$$C_l = \frac{-2(l+\alpha)(l+\beta)}{(l+\alpha+\beta)(2l+\alpha+\beta)(2l+\alpha+\beta+1)}.$$

By (2.11),

$$\gamma_l^{(\alpha,\beta)} \sim \frac{2^{\alpha+\beta+1}}{2l+\alpha+\beta+1}\left(1-\frac{\beta}{l+\alpha+\beta}\right)^{\alpha}\left(1-\frac{\alpha}{l+\alpha+\beta}\right)^{\beta}\left(1+\frac{\alpha\beta}{l+\alpha+\beta}\right)^{l+\frac{1}{2}}.$$

Since $-1<\alpha,\beta<1$, we have $\gamma_l^{(\alpha,\beta)} \sim cl^{-1}$. Furthermore (see [20, p. 249])

$$A_l J_{l+1}^{(\alpha,\beta)}(1) + C_l J_{l-1}^{(\alpha,\beta)}(1) = \frac{2(l+\alpha)\Gamma(l+\alpha)p(l)}{(l+1)!(l+\alpha+\beta)\Gamma(\alpha+1)\ \prod_{k=0}^{2}(2l+\alpha+\beta+k)},$$

where

$$\begin{aligned}p(l) &= (l+\alpha+1)(l+\alpha+\beta)(l+\alpha+\beta+1)(2l+\alpha+\beta)\\ &\quad -\ l(l+1)(l+\beta)(2l+\alpha+\beta+2).\end{aligned}$$

Clearly, for $-1<\alpha,\beta<1, |p(l)|\leqslant cl^3$. Thus by using (2.2) and (2.11), we deduce that

$$|A_l J_{l+1}^{(\alpha,\beta)}(1)+C_l J_{l-1}^{(\alpha,\beta)}(1)|\leqslant c\Gamma^{-1}(\alpha+1)(l+\alpha-1)^{\alpha-2}\left(\frac{l+\alpha-1}{l+1}\right)^{l+\frac{3}{2}} \leqslant c\Gamma^{-1}(\alpha+1)l^{\alpha-2}.$$

Similarly,

$$|A_l J_{l+1}^{(\alpha,\beta)}(-1)+C_l J_{l-1}^{(\alpha,\beta)}(-1)|\leqslant c\Gamma^{-1}(\beta+1)l^{\beta-2}.$$

Moreover,

$$|B_l|\leqslant cl^{-2}, \quad |J_l^{(\alpha,\beta)}(1)|\leqslant c\Gamma^{-1}(\alpha+1)l^{\alpha}, \quad |J_l^{(\alpha,\beta)}(-1)|\leqslant c\Gamma^{-1}(\beta+1)l^{\beta}.$$

The above statements lead to

$$\hat{d}_l\leqslant c\max(\Gamma^{-1}(\alpha+1),\Gamma^{-1}(\beta+1))l^{\max(\alpha,\beta)-1}.$$

Thus

$$\begin{aligned}||\mathcal{T}_{N,\alpha,\beta}1-1||_{\chi^{(-\alpha,-\beta)}} &= \left(\sum_{l=N}^{\infty}\hat{d}_l^2\gamma_l^{(\alpha,\beta)}\right)^{\frac{1}{2}}\\ &\leqslant c\max(\Gamma^{-1}(\alpha+1),\Gamma^{-1}(\beta+1))\left(\sum_{l=N}^{\infty} l^{2\max(\alpha,\beta)-3}\right)^{\frac{1}{2}}\\ &\leqslant c\max(\Gamma^{-1}(\alpha+1),\Gamma^{-1}(\beta+1))\\ &\quad\times\ (N(N+\alpha+\beta))^{\frac{\max(\alpha,\beta)-1}{2}}.\end{aligned} \tag{2.21}$$

We have the same upper-bound for $||\mathcal{T}_{N,\alpha,\beta}x-x||_{\chi^{(-\alpha,-\beta)}}$. So it follows from projection theorem and (2.18) that

$$\begin{aligned}||\mathcal{T}_{N,\alpha,\beta}v-v||_{\chi^{(-\alpha,-\beta)}} &\leqslant ||\mathcal{T}_{N,\alpha,\beta}v^0-v^0||_{\chi^{(-\alpha,-\beta)}}+||\mathcal{T}_{N,\alpha,\beta}v^*-v^*||_{\chi^{(-\alpha,-\beta)}}\\ &\leqslant ||v_N^0-v^0||_{\chi^{(-\alpha,-\beta)}}+\tfrac{1}{2}(|v(1)|+|v(-1)|)(||\mathcal{T}_{N,\alpha,\beta}1-1||_{\chi^{(-\alpha,-\beta)}}\\ &\quad+\ ||\mathcal{T}_{N,\alpha,\beta}x-x||_{\chi^{(-\alpha,-\beta)}}).\end{aligned} \tag{2.22}$$

Finally we obtain the desired result by substituting (2.13), (2.20) and (2.21) into (2.22). If $v\in H^1_{0,\chi^{(-\alpha,-\beta)}}(\Lambda)$, then the second result follows from (2.20) immediately. □

The next lemma will play an important role in the analysis of Jacobi–Gauss-type interpolations.

Lemma 2.2. *There exists a mapping* $\widehat{P}^1_{N,\alpha,\beta}:H^1_{\chi^{(\alpha,\beta)},A}(\Lambda)\to\mathcal{P}_N$ *such that* $\widehat{P}^1_{N,\alpha,\beta}v(-1)=v(-1)$, *and for any* $v\in H^1_{\chi^{(\alpha,\beta)},A}(\Lambda)$,

$$(\partial_x(\widehat{P}^1_{N,\alpha,\beta}v-v),\partial_x\phi)_{\chi^{(\alpha+1,\beta+1)}}=0\quad\forall\phi\in\mathcal{P}_N. \tag{2.23}$$

Moreover, for any $v \in H^r_{\chi^{(\alpha,\beta)},A}(\Lambda)$, $\mu, r \in \mathbb{N}$, $r \geqslant 1$ *and* $0 \leqslant \mu \leqslant r$,

$$||\widehat{P}^1_{N,\alpha,\beta}v - v||_{\mu,\chi^{(\alpha,\beta)},A} \leqslant c\sigma_{\alpha,\beta}(N(N+\alpha+\beta))^{\frac{\mu-r}{2}}|v|_{r,\chi^{(\alpha,\beta)},A}, \tag{2.24}$$

where

$$\sigma_{\alpha,\beta} = \sqrt{2(16\zeta_{\alpha,\beta}\max((\alpha+1)^2,(\beta+1)^2)+1)}.$$

Proof. Let $P_{N,\alpha,\beta}$ be the orthogonal projection as in (2.6), and

$$\widehat{P}^1_{N,\alpha,\beta}v(x) = \int_{-1}^{x} P_{N-1,\alpha+1,\beta+1}\partial_y v(y)\,dy + v(-1). \tag{2.25}$$

Clearly, (2.23) holds and $\widehat{P}^1_{N,\alpha,\beta}v(-1) = v(-1)$. For any integer $\mu \geqslant 1$, we have from Theorem 2.1 that

$$\begin{aligned}|\partial_x^{\mu}(\widehat{P}^1_{N,\alpha,\beta}v - v)|_{\mu,\chi^{(\alpha,\beta)},A} &= ||\partial_x^{\mu-1}(P_{N-1,\alpha+1,\beta+1}\partial_x v - \partial_x v)||_{\chi^{(\alpha+\mu,\beta+\mu)}}\\ &\leqslant c(N(N+\alpha+\beta))^{\frac{\mu-r}{2}}|v|_{r,\chi^{(\alpha,\beta)},A}.\end{aligned} \tag{2.26}$$

We now prove (2.24) with $\mu = 0$ by an duality argument. Let $g \in L^2_{\chi^{(\alpha,\beta)}}(\Lambda)$ and consider an auxiliary problem. It is to find $w \in H^1_{\chi^{(\alpha,\beta)},A}(\Lambda)$ such that

$$(\partial_x w, \partial_x z)_{\chi^{(\alpha+1,\beta+1)}} = (g,z)_{\chi^{(\alpha,\beta)}} \quad \forall z \in H^1_{\chi^{(\alpha,\beta)},A}(\Lambda). \tag{2.27}$$

Let $w(x)$ vary in $\mathcal{D}(\Lambda)$. Then in sense of distributions,

$$-\partial_x(\partial_x w(x)\chi^{(\alpha+1,\beta+1)}(x)) = g(x)\chi^{(\alpha,\beta)}(x). \tag{2.28}$$

Furthermore,

$$-\partial_x^2 w(x) = -((\alpha+\beta+2)x + (\alpha-\beta))(1-x^2)^{-1}\partial_x w(x) + (1-x^2)^{-1}g(x).$$

Thus

$$||\partial_x^2 w||^2_{\chi^{(\alpha+2,\beta+2)}} \leqslant 8\max((\alpha+1)^2,(\beta+1)^2)||\partial_x w||^2_{\chi^{(\alpha,\beta)}} + 2||g||^2_{\chi^{(\alpha,\beta)}}. \tag{2.29}$$

Since $\alpha+1>0$ and $\beta+1>0$, we have that $\partial_x w(x)\chi^{(\alpha+1,\beta+1)}(x) \to 0$ as $|x| \to 1$. By (2.28) and (2.14) with $\psi(x) = g(x)\chi^{(\alpha,\beta)}(x)$ and $q = -\alpha < 1$, we derive that

$$\begin{aligned}&\int_{\Lambda_1}(\partial_x w(x))^2\chi^{(\alpha,\beta)}(x)\,dx\\ &\quad= \int_0^1 \chi^{(-\alpha-2,-\beta-2)}(x)\left(\int_x^1 g(y)\chi^{(\alpha,\beta)}(y)\,dy\right)^2 dx\\ &\quad\leqslant \int_0^1 (1-x)^{-\alpha-2}\left(\int_x^1 g(y)\chi^{(\alpha,\beta)}(y)\,dy\right)^2 dx\\ &\quad\leqslant \frac{4}{\alpha+1}\int_{\Lambda_1} g^2(x)\chi^{(\alpha,2\beta)}(x)\,dx \leqslant \frac{4\xi_\beta}{\alpha+1}\int_{\Lambda_1} g^2(x)\chi^{(\alpha,\beta)}(x)\,dx.\end{aligned}$$

A similar result is valid on Λ_2. Therefore $||\partial_x^2 w||_{\chi^{(\alpha+2,\beta+2)}} \leqslant \sigma_{\alpha,\beta}||g||_{\chi^{(\alpha,\beta)}}$. Now, by taking $z = \widehat{P}^1_{N,\alpha,\beta}v - v$ in (2.27), we use (2.23) and (2.26) to obtain that

$$\begin{aligned} |(\widehat{P}^1_{N,\alpha,\beta}v - v, g)_{\chi^{(\alpha,\beta)}}| &= |(\partial_x(\widehat{P}^1_{N,\alpha,\beta}v - v), P_{N-1,\alpha+1,\beta+1}\partial_x w - \partial_x w)_{\chi^{(\alpha+1,\beta+1)}}| \\ &\leqslant ||\partial_x(\widehat{P}^1_{N,\alpha,\beta}v - v)||_{\chi^{(\alpha+1,\beta+1)}}||P_{N-1,\alpha+1,\beta+1}\partial_x w - \partial_x w||_{\chi^{(\alpha+1,\beta+1)}} \\ &\leqslant c(N(N+\alpha+\beta))^{-\frac{r}{2}}|v|_{r,\chi^{(\alpha,\beta)},\Lambda}||\partial_x^2 w||_{\chi^{(\alpha+2,\beta+2)}} \\ &\leqslant c\sigma_{\alpha,\beta}(N(N+\alpha+\beta))^{-\frac{r}{2}}|v|_{r,\chi^{(\alpha,\beta)},\Lambda}||g||_{\chi^{(\alpha,\beta)}}. \end{aligned}$$

Consequently,

$$\begin{aligned} ||\widehat{P}^1_{N,\alpha,\beta}v - v||_{\chi^{(\alpha,\beta)}} &= \sup_{\substack{g\in L^2_{\chi^{(\alpha,\beta)}}(\Lambda) \\ g\neq 0}} \frac{|(\widehat{P}^1_{N,\alpha,\beta}v - v, g)_{\chi^{(\alpha,\beta)}}|}{||g||_{\chi^{(\alpha,\beta)}}} \\ &\leqslant c\sigma_{\alpha,\beta}(N(N+\alpha+\beta))^{-\frac{r}{2}}|v|_{r,\chi^{(\alpha,\beta)},\Lambda}. \end{aligned} \tag{2.30}$$

Finally, we obtain the desired result with $0<\mu<1$ by using (2.7), (2.26) and (2.30). The desired result for $\mu>1$ comes from (2.7) and (2.26) directly. □

We now establish two embedding inequalities. For simplicity, we set

$$\kappa_{a,b} = \max\left(\frac{\sqrt{\xi_b}}{a+1}, \frac{\sqrt{\xi_a}}{b+1}\right), \quad \Delta_{0,0} = 1, \quad \Delta_{a,b} = \sqrt{2^{a+b}\left(\frac{a}{a+b}\right)^a\left(\frac{b}{a+b}\right)^b}.$$

Lemma 2.3. *If*

$$\alpha\leqslant\gamma+2, \quad \beta\leqslant\delta+2, \tag{2.31}$$

then for any $v\in H^1_{\chi^{(\alpha,\beta)}}(\Lambda)$ *with* $v(x_0) = 0,\ x_0\in\Lambda$,

$$||v||_{\chi^{(\gamma,\delta)}} \leqslant D_{\alpha,\beta,\gamma,\delta}|v|_{1,\chi^{(\alpha,\beta)}}, \tag{2.32}$$

where

$$D_{\alpha,\beta,\gamma,\delta} = 2\Delta_{\gamma-\alpha+2,\delta-\beta+2}\left(\max\left\{\frac{\max(2^\gamma,(1-x_0)^\gamma)}{(\delta+1)^2(1-x_0)^{\gamma+2}}, \frac{\max(2^\delta,(1+x_0)^\delta)}{(\gamma+1)^2(1+x_0)^{\delta+2}}\right\}\right)^{\frac{1}{2}}.$$

In particular, $D_{\alpha,\beta,\gamma,\delta} = \eta^{(1)}_{\alpha,\beta,\gamma,\delta} = 2\kappa_{\gamma,\delta}\Delta_{\gamma-\alpha+2,\delta-\beta+2}$ *for* $x_0 = 0$.

Proof. For any $x\in[x_0,1)$,

$$\begin{aligned}v^2(x)(1-x)^{\gamma+1} &+ (\gamma+1)\int_{x_0}^{x} v^2(y)(1-y)^{\gamma}\,dy\\ &= 2\int_{x_0}^{x} v(y)\partial_y v(y)(1-y)^{\gamma+1}\,dy\\ &\leqslant 2\left(\int_{x_0}^{x} v^2(y)(1-y)^{\gamma}\,dy\right)^{\frac12}\left(\int_{x_0}^{x}(\partial_y v(y))^2(1-y)^{\gamma+2}\,dy\right)^{\frac12}.\end{aligned}$$

Thus

$$\begin{aligned}\int_{x_0}^{1} v^2(x)\chi^{(\gamma,\delta)}(x)\,dx &\leqslant \max(2^{\delta},(1+x_0)^{\delta})\int_{x_0}^{1} v^2(x)(1-x)^{\gamma}\,dx\\ &\leqslant \frac{4\max(2^{\delta},(1+x_0)^{\delta})}{(\gamma+1)^2}\int_{x_0}^{1}(\partial_x v(x))^2(1-x)^{\gamma+2}\,dx\\ &\leqslant \frac{4\max(2^{\delta},(1+x_0)^{\delta})}{(\gamma+1)^2(1+x_0)^{\delta+2}}\int_{x_0}^{1}(\partial_x v(x))^2\chi^{(\gamma+2,\delta+2)}(x)\,dx.\end{aligned}$$

A similar result is valid on the interval $(-1,x_0]$. Moreover,

$$\max_{x\in\bar{\Lambda}}(1-x)^a(1+x)^b=\begin{cases}1 & \text{if } a=b=0,\\ 2^{a+b}\left(\dfrac{a}{a+b}\right)^a\left(\dfrac{b}{a+b}\right)^b & \text{if } a,b\geqslant 0,\ a^2+b^2\neq 0.\end{cases}$$

Therefore,

$$|v|_{1,\chi^{(\gamma+2,\delta+2)}}\leqslant \Delta_{\gamma-\alpha+2,\delta-\beta+2}|v|_{1,\chi^{(\alpha,\beta)}}.$$

Finally, a combination of previous results leads to the desired result. □

Lemma 2.4. *If*

$$\alpha\leqslant\gamma+2,\quad \beta\leqslant 0,\quad \delta\geqslant 0 \tag{2.33}$$

or

$$\alpha\leqslant\gamma+1,\quad \beta\leqslant\delta+2,\quad 0<\alpha<1,\quad \beta<1, \tag{2.34}$$

then for any $v\in H^1_{\chi^{(\alpha,\beta)}}(\Lambda)$ *with* $v(-1)=0$,

$$||v||_{\chi^{(\gamma,\delta)}}\leqslant \eta^{(2)}_{\alpha,\beta,\gamma,\delta}|v|_{1,\chi^{(\alpha,\beta)}}, \tag{2.35}$$

where

$$\eta^{(2)}_{\alpha,\beta,\gamma,\delta}=\begin{cases}2^{\frac{\gamma+\delta-\alpha-\beta}{2}+2}(\gamma+1)^{-1} & \textit{for } (2.33),\\ \sqrt{2}\Delta_{\gamma-\alpha+1,\delta-\beta+2}(\max(4(1-\alpha)^{-1}\xi_{-\beta} & \\ \quad +\alpha^{-1}\gamma_0^{(-\alpha,-\beta)},2(1-\beta)^{-1}\xi_{\alpha-1}))^{\frac12} & \textit{for } (2.34).\end{cases}$$

Proof. We have

$$v^2(x)(1-x)^{\gamma+1} + (\gamma+1)\int_{-1}^{x} v^2(y)(1-y)^{\gamma}\,dy = 2\int_{-1}^{x} v(y)\partial_y v(y)(1-y)^{\gamma+1}\,dy.$$

If (2.33) holds, then

$$\begin{aligned}\int_{\Lambda} v^2(x)(1-x)^{\gamma}\,dx &\leqslant \frac{4}{(\gamma+1)^2}\int_{\Lambda}(\partial_x v(x))^2(1-x)^{\gamma+2}\,dx\\ &\leqslant \frac{2^{\gamma-\alpha+4}}{(\gamma+1)^2}\int_{\Lambda}(\partial_x v(x))^2(1-x)^{\alpha}\,dx\end{aligned}$$

and so

$$\begin{aligned}||v||^2_{\chi^{(\gamma,\delta)}} &\leqslant 2^{\delta}\int_{\Lambda} v^2(x)(1-x)^{\gamma}\,dx \leqslant \frac{2^{\gamma+\delta-\alpha+4}}{(\gamma+1)^2}\int_{\Lambda}(\partial_x v(x))^2(1-x)^{\alpha}\,dx\\ &\leqslant \frac{2^{\gamma+\delta-\alpha-\beta+4}}{(\gamma+1)^2}|v|^2_{1,\chi^{(\alpha,\beta)}}.\end{aligned}$$

Next, we consider the case with (2.34). By (2.14) with $q=\alpha<1$,

$$\int_{\Lambda_1}(v(x)-v(1))^2(1-x)^{\alpha-2}\,dx \leqslant \frac{4\xi_{-\beta}}{1-\alpha}\int_{\Lambda_1}(\partial_x v(x))^2\chi^{(\alpha,\beta)}(x)\,dx. \tag{2.36}$$

Let $|v(x^*)| = \max_{x\in\bar{\Lambda}}|v(x)|$. It can be checked that for $\alpha,\beta<1$,

$$|v(1)| \leqslant |v(x^*)| \leqslant \int_{-1}^{x^*}|\partial_x v(x)|\,dx \leqslant (\gamma_0^{(-\alpha,-\beta)})^{\frac{1}{2}}|v|_{1,\chi^{(\alpha,\beta)}}. \tag{2.37}$$

Furthermore, for $\alpha>0$,

$$\begin{aligned}\int_{\Lambda_1}(v(x)-v(1))^2(1-x)^{\alpha-1}\,dx \geqslant &\int_{\Lambda_1} v^2(x)(1-x)^{\alpha-1}\,dx\\ &+\int_{\Lambda_1}(v^2(1)-2|v(1)|\,|v(x^*)|)(1-x)^{\alpha-1}\,dx\\ = &\int_{\Lambda_1} v^2(x)(1-x)^{\alpha-1}\,dx\\ &+\frac{1}{\alpha}(|v^2(1)|-2|v(1)|\,|v(x^*)|).\end{aligned} \tag{2.38}$$

Therefore we use (2.36)–(2.38) to obtain that for $\beta \leqslant 2$,

$$\begin{aligned}
&\int_{\Lambda_1} v^2(x)\chi^{(\alpha-1,\beta-2)}(x)\,dx \\
&\quad \leqslant \int_{\Lambda_1} v^2(x)(1-x)^{\alpha-1}\,dx \\
&\quad \leqslant \int_{\Lambda_1} (v(x)-v(1))^2(1-x)^{\alpha-1}\,dx + \frac{1}{\alpha}(2|v(1)|\,|v(x^*)| - |v^2(1)|) \\
&\quad \leqslant 2\int_{\Lambda_1} (v(x)-v(1))^2(1-x)^{\alpha-2}\,dx + \frac{2}{\alpha}v^2(x^*) \\
&\quad \leqslant 2\left(\frac{4\xi_{-\beta}}{1-\alpha} + \frac{\gamma_0^{(-\alpha,-\beta)}}{\alpha}\right)|v|^2_{1,\chi^{(\alpha,\beta)}}.
\end{aligned} \tag{2.39}$$

On the other hand, using (2.14) with $q=\beta<1$ yields that

$$\int_{\Lambda_2} v^2(x)(1+x)^{\beta-2}\,dx \leqslant \frac{4}{1-\beta}\int_{\Lambda_2} (\partial_x v(x))^2(1+x)^{\beta}\,dx$$

and so for $\alpha>0$,

$$\begin{aligned}
&\int_{\Lambda_2} v^2(x)\chi^{(\alpha-1,\beta-2)}(x)\,dx \\
&\quad \leqslant \xi_{\alpha-1}\int_{\Lambda_2} v^2(x)(1+x)^{\beta-2}\,dx \\
&\quad \leqslant \frac{4\xi_{\alpha-1}}{1-\beta}\int_{\Lambda_2} (\partial_x v(x))^2(1+x)^{\beta}\,dx \\
&\quad \leqslant \frac{4\xi_{\alpha-1}}{1-\beta}\int_{\Lambda_2} (\partial_x v(x))^2\chi^{(\alpha,\beta)}(x)\,dx.
\end{aligned} \tag{2.40}$$

Finally, a combination of (2.39) and (2.40) leads to

$$||v||^2_{\chi^{(\gamma,\delta)}} \leqslant \Delta^2_{\gamma-\alpha+1,\delta-\beta+1}||v||^2_{\chi^{(\alpha-1,\beta-2)}} \leqslant (\eta^{(2)}_{\alpha,\beta,\gamma,\delta})^2|v|^2_{1,\chi^{(\alpha,\beta)}}. \qquad \square$$

In the end of this section, we present two inverse inequalities.

Lemma 2.5. *For any $\phi \in \mathcal{P}_N$, $r \in \mathbb{N}$ and $\alpha,\beta > r-1$,*

$$||\partial_x^r \phi||_{\chi^{(\alpha,\beta)}} \leqslant (N(N+\alpha+\beta))^{\frac{r}{2}}||\phi||_{\chi^{(\alpha-r,\beta-r)}}. \tag{2.41}$$

Proof. Let $r=1, \alpha,\beta>0$ and $\hat{\phi}_l^{(\alpha-1,\beta-1)}$ be the Jacobi-coefficients of $\phi(x)$ in terms of $J_l^{(\alpha-1,\beta-1)}(x)$. By (2.3),

$$||\partial_x \phi||^2_{\chi^{(\alpha,\beta)}} = \frac{1}{4}\sum_{l=0}^{N-1}(l+\alpha+\beta)^2\gamma_l^{(\alpha,\beta)}(\hat{\phi}_{l+1}^{(\alpha-1,\beta-1)})^2. \tag{2.42}$$

Using (2.5) and $\Gamma(x+1) = x\Gamma(x)$, we deduce that

$$\frac{1}{4}\max_{0\leqslant l\leqslant N-1}(l+\alpha+\beta)^2\gamma_l^{(\alpha,\beta)}(\gamma_{l+1}^{(\alpha-1,\beta-1)})^{-1} = N(N+\alpha+\beta-1).$$

Therefore, we use (2.42) to reach that

$$||\partial_x\phi||_{\chi^{(\alpha,\beta)}} \leqslant (N(N+\alpha+\beta))^{\frac{1}{2}}||\phi||_{\chi^{(\alpha-1,\beta-1)}}.$$

Repeating the above procedure leads to the desired result. □

Lemma 2.6. *If one of the following conditions holds*:

$$\text{(i)}\quad \alpha=\beta\geqslant -\frac{1}{2},\quad \text{(ii)}\quad \alpha\geqslant\beta+1,\quad \beta\geqslant 0,\quad \text{(iii)}\quad \frac{1}{2}\leqslant\alpha\leqslant\beta+1, \tag{2.43}$$

then for any $\phi\in\mathcal{P}_N$,

$$||\partial_x\phi||_{\chi^{(\alpha,\beta)}} \leqslant c(\alpha-\beta+1)N(N+\alpha+\beta)||\phi||_{\chi^{(\alpha,\beta)}}. \tag{2.44}$$

Proof. Let $\hat{\phi}_l^{(\alpha,\beta)}$ be the Jacobi-coefficients of $\phi(x)$ in terms of $J_l^{(\alpha,\beta)}(x)$. For simplicity, we introduce the following notations:

$$E_l = \frac{\Gamma(l+\beta+2)}{\Gamma(l+\alpha+\beta+2)}, \quad G_k = \frac{(2k+\alpha+\beta+2)\Gamma(k+\alpha+1)}{\Gamma(k+\beta+2)},$$
$$H_j = \frac{(2j+\alpha+\beta+1)\Gamma(j+\alpha+\beta+1)}{\Gamma(j+\alpha+1)}, \quad \psi_{j,l} = \sum_{k=j}^{l}(-1)^k G_k. \tag{2.45}$$

It is shown on p. 378 of [17] that

$$\partial_x\phi(x) = \frac{1}{2}\sum_{j=0}^{N-1}(-1)^j H_j J_j^{(\alpha,\beta)}(x)\left(\sum_{l=j}^{N-1}E_l\psi_{j,l}\hat{\phi}_{l+1}^{(\alpha,\beta)}\right). \tag{2.46}$$

Thus by (2.4),

$$\begin{aligned}||\partial_x\phi||^2_{\chi^{(\alpha,\beta)}} &\leqslant \frac{1}{4}\sum_{j=0}^{N-1}H_j^2\gamma_j^{(\alpha,\beta)}\left(\sum_{l=j}^{N-1}E_l^2\psi_{j,l}^2(\gamma_{l+1}^{(\alpha,\beta)})^{-1}\right)\left(\sum_{l=j}^{N-1}\gamma_{l+1}^{(\alpha,\beta)}(\hat{\phi}_{l+1}^{(\alpha,\beta)})^2\right)\\ &\leqslant M_{\alpha,\beta,N}||\phi||^2_{\chi^{(\alpha,\beta)}},\end{aligned} \tag{2.47}$$

where

$$M_{\alpha,\beta,N} = \frac{1}{4}\sum_{j=0}^{N-1}H_j^2\gamma_j^{(\alpha,\beta)}\left(\sum_{l=j}^{N-1}E_l^2\psi_{j,l}^2(\gamma_{l+1}^{(\alpha,\beta)})^{-1}\right).$$

Moreover, using (2.5) and (2.45) yields that

$$H_j^2\gamma_j^{(\alpha,\beta)} = \frac{2^{\alpha+\beta+1}(2j+\alpha+\beta+1)\Gamma(j+\alpha+\beta+1)\Gamma(j+\beta+1)}{\Gamma(j+1)\Gamma(j+\alpha+1)} \tag{2.48}$$

and

$$E_l^2(\gamma_{l+1}^{(\alpha,\beta)})^{-1} = \frac{(2l+\alpha+\beta+3)\Gamma(l+2)\Gamma(l+\beta+2)}{2^{\alpha+\beta+1}\Gamma(l+\alpha+2)\Gamma(l+\alpha+\beta+2)}. \tag{2.49}$$

If $\alpha=\beta$, then $G_k=2$ and so $|\psi_{j,l}|\leqslant 2$. Thus by (2.5), (2.45) and (2.47),

$$\begin{aligned}||\partial_x\phi||^2_{\chi^{(\alpha,\alpha)}} \leqslant \Bigg(&\sum_{j=0}^{N-1}(2j+2\alpha+1)\frac{\Gamma(j+2\alpha+1)}{\Gamma(j+1)} \\ &\times\sum_{l=j}^{N-1}(l+1)(2l+2\alpha+3)\frac{\Gamma(l+1)}{\Gamma(l+2\alpha+2)}\Bigg)||\phi||^2_{\chi^{(\alpha,\alpha)}}.\end{aligned} \tag{2.50}$$

If, in addition, $\alpha\geqslant-\frac{1}{2}$, then $\Gamma(l+1)\Gamma^{-1}(l+2\alpha+2)$ decreases as l increases. Hence

$$\max_{j\leqslant l\leqslant N-1}\Gamma(l+1)(\Gamma(l+2\alpha+2))^{-1} = \Gamma(j+1)(\Gamma(j+2\alpha+2))^{-1}.$$

Using the above, we obtain from (2.50) that

$$\begin{aligned}||\partial_x\phi||^2_{\chi^{(\alpha,\alpha)}} &\leqslant \left(\sum_{j=0}^{N-1}\frac{2j+2\alpha+1}{j+2\alpha+1}\sum_{l=j}^{N-1}(l+1)(2l+2\alpha+3)\right)||\phi||^2_{\chi^{(\alpha,\alpha)}} \\ &\leqslant 8N^2(N+\alpha)^2||\phi||^2_{\chi^{(\alpha,\alpha)}}.\end{aligned} \tag{2.51}$$

This implies (2.44) with $\alpha=\beta\geqslant-\frac{1}{2}$.

Next, we consider the case of $\alpha\geqslant\beta+1$ and $\beta\geqslant 0$. Let

$$A_k = (2k+\alpha+\beta+3)\Gamma(k+\alpha+1)(\Gamma(k+\beta+3))^{-1}.$$

It can be checked that $G_{k+1}-G_k=(\alpha-\beta)A_k$ and

$$A_{k+1}-A_k = (\alpha-\beta-1)(2k+\alpha+\beta+4)\Gamma(k+\alpha+1)(\Gamma(k+\beta+4))^{-1}. \tag{2.52}$$

Thus for $\alpha\geqslant\beta+1$,

$$|\psi_{j,l}|\leqslant(\alpha-\beta)\sum_{k=j}^{l}A_k\leqslant(\alpha-\beta)(l-j)A_l. \tag{2.53}$$

Let

$$T_l = \Gamma(l+1)\Gamma(l+\alpha+1)(\Gamma(l+\beta+1)\Gamma(l+\alpha+\beta+1))^{-1}.$$

Then (2.49) with (2.53) leads to that

$$\begin{aligned}E_l^2\psi_{l,j}^2(\gamma_{l+1}^{(\alpha,\beta)})^{-1} &\leqslant (\alpha-\beta)^2(l-j)^2E_l^2A_l^2(\gamma_{l+1}^{(\alpha,\beta)})^{-1} \\ &\leqslant c(\alpha-\beta)^2 2^{-(\alpha+\beta+1)}(2l+\alpha+\beta+3)T_l.\end{aligned} \tag{2.54}$$

Clearly, $T_l \geqslant T_{l+1}$ for $\beta \geqslant 0$. Thus using (2.48) and (2.54) gives that

$$\begin{aligned} 4M_{\alpha,\beta,N} &\leqslant c(\alpha-\beta)^2 2^{-(\alpha+\beta+1)} N(N+\alpha+\beta+1) \sum_{j=0}^{N-1} H_j^2 \gamma_j^{(\alpha,\beta)} T_j \\ &\leqslant c(\alpha-\beta)^2 N(N+\alpha+\beta+1) \sum_{j=0}^{N-1} (j+\alpha+\beta+1) \\ &\leqslant c(\alpha-\beta)^2 N^2 (N+\alpha+\beta)^2. \end{aligned} \tag{2.55}$$

A combination of (2.47) and (2.55) leads to (2.44) with $\alpha \geqslant \beta+1$ and $\beta \geqslant 0$.

Finally, we prove (2.44) with $\frac{1}{2} \leqslant \alpha \leqslant \beta+1$. We have from (2.52) that

$$|\psi_{j,l}| \leqslant |\alpha-\beta| \sum_{k=j}^{l} A_k \leqslant |\alpha-\beta|(l-j)A_j. \tag{2.56}$$

By (2.48),

$$H_j^2 \gamma_j^{(\alpha,\beta)} A_j^2 = \frac{2^{\alpha+\beta+1}(2j+\alpha+\beta+1)(2j+\alpha+\beta+3)^2}{(j+\beta+1)(j+\beta+2)^2} S_j \leqslant c 2^{\alpha+\beta+1} S_j, \tag{2.57}$$

where

$$S_j = \Gamma(j+\alpha+1)\Gamma(j+\alpha+\beta+1)(\Gamma(j+1)\Gamma(j+\beta+2))^{-1}.$$

It can checked that $S_{j+1} \geqslant S_j$ for $\alpha \geqslant \frac{1}{2}$. Therefore, by (2.49) and (2.57),

$$\begin{aligned} 4M_{\alpha,\beta,N} &= \sum_{l=0}^{N-1} E_l^2 (\gamma_{l+1}^{(\alpha,\beta)})^{-1} \left(\sum_{j=0}^{l} H_j^2 \psi_{j,l}^2 \gamma_j^{(\alpha,\beta)} \right) \\ &\leqslant c(\alpha-\beta)^2 2^{\alpha+\beta+1} \sum_{l=0}^{N-1} l^3 E_l^2 (\gamma_{l+1}^{(\alpha,\beta)})^{-1} S_l \leqslant c(\alpha-\beta)^2 N^4. \end{aligned}$$

This implies the desired result. $\square$

By Theorem 2.2 of [17], for any $\phi \in \mathcal{P}_N, r \in \mathbb{N}$ and $\alpha, \beta > -1$, $||\partial_x \phi||_{\chi^{(\alpha,\beta)}} \leqslant c(\alpha,\beta) N^2 ||\phi||_{\chi^{(\alpha,\beta)}}$ where $c(\alpha,\beta)$ is a constant depending only on α and β.

3. Orthogonal projections in non-uniformly Jacobi-weighted Sobolev spaces

In many practical problems, the coefficients of terms involving derivatives of different orders degenerate in different ways, such as singular differential equations, differential equations in unbounded domains and axisymmetric domains. In these cases, the exact solutions are not in the usual Sobolev spaces, but in non-uniformly Jacobi-weighted Sobolev spaces. In this section, we consider Jacobi approximations in such spaces.

Let $\alpha,\beta,\gamma,\delta > -1$. We introduce the space $H^{\mu}_{\alpha,\beta,\gamma,\delta}(\Lambda), 0\leqslant\mu\leqslant 1$. For $\mu = 0, H^{0}_{\alpha,\beta,\gamma,\delta}(\Lambda) = L^2_{\chi^{(\gamma,\delta)}}(\Lambda)$. For $\mu = 1$,

$$H^{1}_{\alpha,\beta,\gamma,\delta}(\Lambda) = \{v \mid v \text{ is measurable and } ||v||_{1,\alpha,\beta,\gamma,\delta} < \infty\}$$

equipped with the norm

$$||v||_{1,\alpha,\beta,\gamma,\delta} = (|v|^2_{1,\chi^{(\alpha,\beta)}} + ||v||^2_{\chi^{(\gamma,\delta)}})^{\frac{1}{2}}.$$

For $0<\mu<1$, the space $H^{\mu}_{\alpha,\beta,\gamma,\delta}(\Lambda)$ is defined by complex interpolation as in [4]. In other words, $H^{\mu}_{\alpha,\beta,\gamma,\delta}(\Lambda) = [H^{1}_{\alpha,\beta,\gamma,\delta}(\Lambda), L^2_{\chi^{(\gamma,\delta)}}(\Lambda)]_{1-\mu}$. Its norm is denoted by $||v||_{\mu,\alpha,\beta,\gamma,\delta}$. Moreover for any $v\in H^{1}_{\alpha,\beta,\gamma,\delta}(\Lambda)$,

$$||v||_{\mu,\alpha,\beta,\gamma,\delta} \leqslant ||v||^{\mu}_{1,\alpha,\beta,\gamma,\delta} ||v||^{1-\mu}_{\chi^{(\gamma,\delta)}}. \tag{3.1}$$

We also define the spaces

$${}_0H^{1}_{\alpha,\beta,\gamma,\delta}(\Lambda) = \{v \mid v\in H^{1}_{\alpha,\beta,\gamma,\delta}(\Lambda) \text{ and } v(-1) = 0\},$$

$$H^{1}_{0,\alpha,\beta,\gamma,\delta}(\Lambda) = \{v \mid v\in H^{1}_{\alpha,\beta,\gamma,\delta}(\Lambda) \text{ and } v(-1) = v(1) = 0\}.$$

Now, let

$$a_{\alpha,\beta,\gamma,\delta}(u,v) = (\partial_x u, \partial_x v)_{\chi^{(\alpha,\beta)}} + (u,v)_{\chi^{(\gamma,\delta)}} \quad \forall u,v\in H^{1}_{\alpha,\beta,\gamma,\delta}(\Lambda).$$

The orthogonal projection $P^1_{N,\alpha,\beta,\gamma,\delta} : H^{1}_{\alpha,\beta,\gamma,\delta}(\Lambda) \to \mathcal{P}_N$ is defined by

$$a_{\alpha,\beta,\gamma,\delta}(P^1_{N,\alpha,\beta,\gamma,\delta} v - v, \phi) = 0 \quad \forall \phi\in\mathcal{P}_N.$$

For description of approximation results, we introduce the space $H^r_{\chi^{(\alpha,\beta)},*}(\Lambda),\ r\geqslant 1$, with the following weighted norm and semi-norm,

$$||v||_{r,\chi^{(\alpha,\beta)},*} = \left(\sum_{k=0}^{r-1} ||\partial_x^{k+1} v||^2_{\chi^{(\alpha+k,\beta+k)}}\right)^{\frac{1}{2}}, \quad |v|_{r,\chi^{(\alpha,\beta)},*} = ||\partial_x^r v||_{\chi^{(\alpha+r-1,\beta+r-1)}}.$$

Theorem 3.1. *If* (2.31) *holds, then for any* $v\in H^{1}_{\alpha,\beta,\gamma,\delta}(\Lambda)\cap H^r_{\chi^{(\alpha,\beta)},*}(\Lambda)$, $r\in\mathbb{N}$ *and* $r\geqslant 1$,

$$||P^1_{N,\alpha,\beta,\gamma,\delta} v - v||_{1,\alpha,\beta,\gamma,\delta} \leqslant c(\eta^{(1)}_{\alpha,\beta,\gamma,\delta} + 1)(N(N+\alpha+\beta))^{\frac{1-r}{2}} |v|_{r,\chi^{(\alpha,\beta)},*}. \tag{3.2}$$

If, in addition,

$$\alpha\leqslant\gamma+1, \quad \beta\leqslant\delta+1, \tag{3.3}$$

then for $0\leqslant\mu\leqslant 1$,

$$||P^1_{N,\alpha,\beta,\gamma,\delta} v - v||_{\mu,\alpha,\beta,\gamma,\delta} \leqslant c C^{1,\mu}_{\alpha,\beta,\gamma,\delta} (N(N+\alpha+\beta))^{\frac{\mu-r}{2}} |v|_{r,\chi^{(\alpha,\beta)},*} \tag{3.4}$$

where $C^{1,\mu}_{\alpha,\beta,\gamma,\delta} = (\eta^{(1)}_{\alpha,\beta,\gamma,\delta} + 1)^{\mu} (C^{1,0}_{\alpha,\beta,\gamma,\delta})^{1-\mu}$ *and*

$$C^{1,0}_{\alpha,\beta,\gamma,\delta} = 2(\eta^{(1)}_{\alpha,\beta,\gamma,\delta} + 1)^2 (4\zeta^2_{\gamma,\delta}(\alpha^2+\beta^2) + \Delta^2_{\gamma-\alpha+1,\delta-\beta+1})^{\frac{1}{2}}.$$

Proof. Let

$$\phi(x) = \int_{-1}^{x} P_{N-1,\alpha,\beta}\partial_y v(y)\, dy + \xi,$$

where ξ is chosen in such a way that $v(0) = \phi(0)$. By projection theorem, Theorem 2.1 and Lemma 2.3,

$$\begin{aligned}||P^1_{N,\alpha,\beta,\gamma,\delta}v - v||_{1,\alpha,\beta,\gamma,\delta} &\leqslant ||\phi - v||_{1,\alpha,\beta,\gamma,\delta} \leqslant (\eta^{(1)}_{\alpha,\beta,\gamma,\delta} + 1)|\phi - v|_{1,\chi^{(\alpha,\beta)}} \\ &= (\eta^{(1)}_{\alpha,\beta,\gamma,\delta} + 1)||P_{N-1,\alpha,\beta}\partial_x v - \partial_x v||_{\chi^{(\alpha,\beta)}} \\ &\leqslant c(\eta^{(1)}_{\alpha,\beta,\gamma,\delta} + 1)(N(N+\alpha+\beta))^{\frac{1-r}{2}}|v|_{r,\chi^{(\alpha,\beta)},*}. \end{aligned} \tag{3.5}$$

We next consider the case with (3.3). Let $g \in L^2_{\chi^{(\gamma,\delta)}}(\Lambda)$ and consider the auxiliary problem

$$a_{\alpha,\beta,\gamma,\delta}(w,z) = (g,z)_{\chi^{(\gamma,\delta)}} \quad \forall z \in H^1_{\alpha,\beta,\gamma,\delta}(\Lambda). \tag{3.6}$$

Taking $z = w$ in (3.6), we get that $||w||_{1,\alpha,\beta,\gamma,\delta} \leqslant ||g||_{\chi^{(\gamma,\delta)}}$. Let $w(x)$ vary in $\mathcal{D}(\Lambda)$. Then in sense of distributions,

$$-\partial_x(\partial_x w(x)\chi^{(\alpha,\beta)}(x)) = (g(x) - w(x))\chi^{(\gamma,\delta)}(x). \tag{3.7}$$

It can be verified as in the proof of Theorem 2.5 of [17] that $\partial_x w(x)\chi^{(\alpha,\beta)}(x) \to 0$ as $|x| \to 1$. Moreover by (3.7),

$$\begin{aligned}-\partial_x^2 w(x) = &-((\alpha+\beta)x + (\alpha-\beta))(1-x^2)^{-1}\partial_x w(x) \\ &+ (g(x) - w(x))\chi^{(\gamma-\alpha,\beta-\delta)}(x).\end{aligned} \tag{3.8}$$

It is not difficult to show that

$$||\partial_x^2 w(1-x^2)^{\frac{1}{2}}||^2_{\chi^{(\alpha,\beta)}} \leqslant D_1 + D_2, \tag{3.9}$$

where $D_1 = D_1(\Lambda_1) + D_1(\Lambda_2)$ and

$$D_1(\Lambda_j) = 8(\alpha^2+\beta^2)\int_{\Lambda_j}(\partial_x w(x))^2\chi^{(\alpha-1,\beta-1)}(x)\,dx, \quad j = 1,2,$$

$$D_2 = 2\left|\int_{\Lambda}(g(x) - w(x))^2\chi^{(2\gamma-\alpha+1,2\delta-\beta+1)}(x)\,dx\right|.$$

Obviously, (3.3) implies that

$$D_2 \leqslant 2\Delta^2_{\gamma-\alpha+1,\delta-\beta+1}||g - w||^2_{\chi^{(\gamma,\delta)}} \leqslant 4\Delta^2_{\gamma-\alpha+1,\delta-\beta+1}||g||^2_{\chi^{(\gamma,\delta)}}.$$

Thus it remains to estimate $D_1(\Lambda_j)$. By (3.3), (3.7) and (2.14) with $q=-\gamma$,

$$\begin{aligned}
&D_1(\Lambda_1)\\
&\quad = 8(\alpha^2+\beta^2)\int_{\Lambda_1}(1-x)^{-\alpha-1}(1+x)^{-\beta-1}\left(\int_x^1 (g(y)-w(y))\chi^{(\gamma,\delta)}(y)\,dy\right)^2 dx\\
&\quad \leqslant 8(\alpha^2+\beta^2)\int_{\Lambda_1}(1-x)^{-\gamma}\left(\frac{1}{1-x}\int_x^1 (g(y)-w(y))\chi^{(\gamma,\delta)}(y)\,dy\right)^2 dx\\
&\quad \leqslant 8\xi_\delta(\alpha^2+\beta^2)(\gamma+1)^{-1}\int_{\Lambda_1}(g(x)-w(x))^2\chi^{(\gamma,\delta)}(x)\,dx.
\end{aligned}$$

We can estimate $D_1(\Lambda_2)$ similarly. Therefore we obtain from (3.9) that

$$||\partial_x^2 w(1-x^2)^{\frac{1}{2}}||^2_{\chi^{(\alpha,\beta)}}\leqslant 4(4\zeta^2_{\gamma,\delta}(\alpha^2+\beta^2)+\Delta^2_{\gamma-\alpha+1,\delta-\beta+1})||g||^2_{\chi^{(\gamma,\delta)}}. \tag{3.10}$$

Furthermore, using (3.5) and (3.9) yields that

$$\begin{aligned}
&||P^1_{N,\alpha,\beta,\gamma,\delta}w-w||_{1,\alpha,\beta,\gamma,\delta}\\
&\quad\leqslant c(\eta^{(1)}_{\alpha,\beta,\gamma,\delta}+1)(N(N+\alpha+\beta))^{-\frac{1}{2}}||\partial_x^2 w(1-x^2)^{\frac{1}{2}}||_{\chi^{(\alpha,\beta)}}\\
&\quad\leqslant c(\eta^{(1)}_{\alpha,\beta,\gamma,\delta}+1)(4\zeta^2_{\gamma,\delta}(\alpha^2+\beta^2)\\
&\qquad +\ \Delta^2_{\gamma-\alpha+1,\delta-\beta+1})^{\frac{1}{2}}(N(N+\alpha+\beta))^{-\frac{1}{2}}||g||_{\chi^{(\gamma,\delta)}}.
\end{aligned} \tag{3.11}$$

Now, taking $z=P^1_{N,\alpha,\beta,\gamma,\delta}v-v$ in (3.6), we use (3.5) and (3.11) to verify that

$$\begin{aligned}
|(P^1_{N,\alpha,\beta,\gamma,\delta}v-v,g)_{\chi^{(\gamma,\delta)}}| &= |a_{\alpha,\beta,\gamma,\delta}(P^1_{N,\alpha,\beta,\gamma,\delta}v-v,P^1_{N,\alpha,\beta,\gamma,\delta}w-w)|\\
&\leqslant cC^{1,0}_{\alpha,\beta,\gamma,\delta}(N(N+\alpha+\beta))^{-\frac{r}{2}}||g||_{\chi^{(\gamma,\delta)}}|v|_{r,\chi^{(\alpha,\beta)},*}.
\end{aligned}$$

Consequently,

$$||P^1_{N,\alpha,\beta,\gamma,\delta}v-v||_{\chi^{(\gamma,\delta)}}\leqslant cC^{1,0}_{\alpha,\beta,\gamma,\delta}(N(N+\alpha+\beta+1))^{-\frac{r}{2}}|v|_{r,\chi^{(\alpha,\beta)},*}. \tag{3.12}$$

Finally the result for $0<\mu<1$ follows from (3.1), (3.5) and (3.12). $\square$

In some cases, we have to study the orthogonal projection ${}_0P^1_{N,\alpha,\beta,\gamma,\delta}$: ${}_0H^1_{\alpha,\beta,\gamma,\delta}(\Lambda)\to {}_0\mathcal{P}_N$, defined by,

$$a_{\alpha,\beta,\gamma,\delta}({}_0P^1_{N,\alpha,\beta,\gamma,\delta}v-v,\phi)=0\quad \forall\phi\in{}_0\mathcal{P}_N.$$

Theorem 3.2. *If* (2.33) *or* (2.34) *holds, then for any* $v\in{}_0H^1_{\alpha,\beta,\gamma,\delta}(\Lambda)\cap H^r_{\chi^{(\alpha,\beta)},*}(\Lambda)$, $r\in\mathbb{N}$ *and* $r\geqslant 1$,

$$||{}_0P^1_{N,\alpha,\beta,\gamma,\delta}v-v||_{1,\alpha,\beta,\gamma,\delta}\leqslant c(\eta^{(2)}_{\alpha,\beta,\gamma,\delta}+1)(N(N+\alpha+\beta))^{\frac{1-r}{2}}|v|_{r,\chi^{(\alpha,\beta)},*}. \tag{3.13}$$

If, in addition, (3.3) *holds, then for* $0\leqslant\mu\leqslant 1$,

$$||{}_0P^1_{N,\alpha,\beta,\gamma,\delta}v-v||_{\mu,\alpha,\beta,\gamma,\delta}\leqslant cC^{2,\mu}_{\alpha,\beta,\gamma,\delta}(N(N+\alpha+\beta))^{\frac{\mu-r}{2}}|v|_{r,\chi^{(\alpha,\beta)},*}, \tag{3.14}$$

where $C^{2,\mu}_{\alpha,\beta,\gamma,\delta} = (\eta^{(2)}_{\alpha,\beta,\gamma,\delta} + 1)^{\mu} (C^{2,0}_{\alpha,\beta,\gamma,\delta})^{1-\mu}$ *and*

$$C^{2,0}_{\alpha,\beta,\gamma,\delta} = 2(\eta^{(2)}_{\alpha,\beta,\gamma,\delta} + 1)^2 (4\zeta^2_{\gamma,\delta}(\alpha^2 + \beta^2) + \Delta^2_{\gamma-\alpha+1,\delta-\beta+1})^{\frac{1}{2}}.$$

Proof. Let

$$\phi(x) = \int_{-1}^{x} P_{N-1,\alpha,\beta} \partial_y v(y)\, dy.$$

By projection theorem, Lemma 2.4 and Theorem 2.1,

$$\begin{aligned} \|{}_0P^1_{N,\alpha,\beta,\gamma,\delta} v - v\|_{1,\alpha,\beta,\gamma,\delta} &\leqslant \|\phi - v\|_{1,\alpha,\beta,\gamma,\delta} \leqslant (\eta^{(2)}_{\alpha,\beta,\gamma,\delta} + 1)|\phi - v|_{1,\chi^{(\alpha,\beta)}} \\ &= (\eta^{(2)}_{\alpha,\beta,\gamma,\delta} + 1)\|P_{N-1,\alpha,\beta}\partial_x v - \partial_x v\|_{\chi^{(\alpha,\beta)}} \\ &\leqslant c(\eta^{(2)}_{\alpha,\beta,\gamma,\delta} + 1)(N(N+\alpha+\beta))^{\frac{1-r}{2}} |v|_{r,\chi^{(\alpha,\beta)},*}. \end{aligned} \tag{3.15}$$

Now, let (3.3) hold and $g \in L^2_{\chi^{(\gamma,\delta)}}(\Lambda)$. We consider the auxiliary problem

$$a_{\alpha,\beta,\gamma,\delta}(w,z) = (g,z)_{\chi^{(\gamma,\delta)}} \quad \forall z \in {}_0H^1_{\alpha,\beta,\gamma,\delta}(\Lambda). \tag{3.16}$$

Taking $z = w$ in (3.16), we get that $\|w\|_{1,\alpha,\beta,\gamma,\delta} \leqslant c\|g\|_{\chi^{(\gamma,\delta)}}$. It can be shown that (3.7) still holds and $\partial_x w(x)\chi^{(\alpha,\beta)}(x) \to 0$ as $x \to -1$. Moreover (3.8) and (3.9) are valid also. We can estimate D_1 and D_2 in (3.9) as in the proof of Theorem 3.1. Finally, a duality argument and (3.1) lead to the desired result. □

There are several $H^1_{0,\alpha,\beta,\gamma,\delta}(\Lambda)$-orthogonal projections corresponding to various practical problems. The orthogonal projection $P^{1,0}_{N,\alpha,\beta,\gamma,\delta} : H^1_{0,\alpha,\beta,\gamma,\delta}(\Lambda) \to \mathcal{P}^0_N$ is defined by,

$$a_{\alpha,\beta,\gamma,\delta}(P^{1,0}_{N,\alpha,\beta,\gamma,\delta} v - v, \phi) = 0 \quad \forall \phi \in \mathcal{P}^0_N.$$

Theorem 3.3. *Let* $\gamma \leqslant \alpha \leqslant \gamma + 1$, $\delta \leqslant \beta \leqslant \delta + 1$ *and* $\gamma, \delta < 1$. *If for* $r \in \mathbb{N}$, $r \geqslant 2$, $v \in H^1_{0,\alpha,\beta,\gamma,\delta}(\Lambda)$ *and* $\partial_x v \in H^{r-1}_{\chi^{(\alpha,\beta)},*}(\Lambda)$, *then*

$$\|P^{1,0}_{N,\alpha,\beta,\gamma,\delta} v - v\|_{1,\alpha,\beta,\gamma,\delta} \leqslant c C^{3,1}_{\alpha,\beta,\gamma,\delta} (N(N+\alpha+\beta))^{\frac{1-r}{2}} |\partial_x v|_{r-1,\chi^{(\alpha,\beta)},*}, \tag{3.17}$$

where

$$C^{3,1}_{\alpha,\beta,\gamma,\delta} = C^{1,0}_{\alpha,\beta,\gamma,\delta}(\Delta_{\alpha-\gamma,\beta-\delta} + \tfrac{1}{2}(\gamma_0^{(-\gamma,-\delta)})^{\frac{1}{2}}((\gamma_0^{(\alpha,\beta)})^{\frac{1}{2}} + (\gamma_0^{(\gamma,\delta+2)})^{\frac{1}{2}} + 2(\gamma_0^{(\gamma,\delta)})^{\frac{1}{2}})).$$

Proof. Let

$$\phi^*(x) = \int_{-1}^{x} P^1_{N-1,\alpha,\beta,\gamma,\delta} \partial_y v(y)\, dy, \quad \phi(x) = \phi^*(x) - \tfrac{1}{2}\phi^*(1)(x+1).$$

Clearly $\phi \in \mathcal{P}_N^0$. Since $\gamma \leqslant \alpha$ and $\delta \leqslant \beta$, we have from projection theorem that

$$\begin{aligned}
&||P^{1,0}_{N,\alpha,\beta,\gamma,\delta}v - v||_{1,\alpha,\beta,\gamma,\delta} \\
&\quad \leqslant ||\phi - v||_{1,\alpha,\beta,\gamma,\delta} \\
&\quad \leqslant ||P^1_{N-1,\alpha,\beta,\gamma,\delta}\partial_x v - \partial_x v||_{\chi^{(\alpha,\beta)}} + \tfrac{1}{2}(\gamma_0^{(\alpha,\beta)})^{\frac{1}{2}}|\phi^*(1)| + ||\phi - v||_{\chi^{(\gamma,\delta)}} \\
&\quad \leqslant \Delta_{\alpha-\gamma,\beta-\delta}||P^1_{N-1,\alpha,\beta,\gamma,\delta}\partial_x v - \partial_x v||_{\chi^{(\gamma,\delta)}} + \tfrac{1}{2}(\gamma_0^{(\alpha,\beta)})^{\frac{1}{2}}|\phi^*(1)| + ||\phi - v||_{\chi^{(\gamma,\delta)}}. \quad (3.18)
\end{aligned}$$

Next, thanks to $\gamma, \delta < 1$,

$$\begin{aligned}
|\phi^*(1)| &= \left| \int_\Lambda (P^1_{N-1,\alpha,\beta,\gamma,\delta}\partial_x v(x) - \partial_x v(x))\, dx \right| \\
&\leqslant (\gamma_0^{(-\gamma,-\delta)})^{\frac{1}{2}}||P^1_{N-1,\alpha,\beta,\gamma,\delta}\partial_x v - \partial_x v||_{\chi^{(\gamma,\delta)}}. \quad (3.19)
\end{aligned}$$

Moreover, we use the above result to verify that

$$\begin{aligned}
||\phi - v||_{\chi^{(\gamma,\delta)}} &\leqslant (\gamma_0^{(\gamma,\delta)}\gamma_0^{(-\gamma,-\delta)})^{\frac{1}{2}}||P^1_{N-1,\alpha,\beta,\gamma,\delta}\partial_x v - \partial_x v||_{\chi^{(\gamma,\delta)}} + \tfrac{1}{2}(\gamma_0^{(\gamma,\delta+2)})^{\frac{1}{2}}|\phi^*(1)| \\
&\leqslant (\tfrac{1}{2}(\gamma_0^{(\gamma,\delta+2)}\gamma_0^{(-\gamma,-\delta)})^{\frac{1}{2}} + (\gamma_0^{(\gamma,\delta)}\gamma_0^{(-\gamma,-\delta)})^{\frac{1}{2}}) \\
&\quad \times ||P^1_{N-1,\alpha,\beta,\gamma,\delta}\partial_x v - \partial_x v||_{\chi^{(\gamma,\delta)}}. \quad (3.20)
\end{aligned}$$

Substituting (3.19) and (3.20) into (3.18) and using Theorem 3.1, we obtain (3.17). □

We now turn to another orthogonal projection. Let

$$\widetilde{a}_{\alpha,\beta}(u,v) = (\partial_x u, \partial_x v)_{\chi^{(\alpha,\beta)}}.$$

The orthogonal projection $\widetilde{P}^{1,0}_{N,\alpha,\beta} : H^1_{0,\chi^{(\alpha,\beta)}}(\Lambda) \to \mathcal{P}^0_N$ is defined by

$$\widetilde{a}_{\alpha,\beta}(\widetilde{P}^{1,0}_{N,\alpha,\beta}v - v, \phi) = 0 \quad \forall \phi \in \mathcal{P}^0_N. \quad (3.21)$$

Theorem 3.4. *If* $-1 < \alpha, \beta < 1$, *then for any* $v \in H^1_{0,\chi^{(\alpha,\beta)}}(\Lambda) \cap H^r_{\chi^{(\alpha,\beta)},*}(\Lambda)$, $r \in \mathbb{N}$ *and* $r \geqslant 1$,

$$||\widetilde{P}^{1,0}_{N,\alpha,\beta}v - v||_{1,\chi^{(\alpha,\beta)}} \leqslant c\widetilde{C}^{3,1}_{\alpha,\beta}(N(N+\alpha+\beta))^{\frac{1-r}{2}}|v|_{r,\chi^{(\alpha,\beta)},*}, \quad (3.22)$$

where

$$\widetilde{C}^{3,1}_{\alpha,\beta} = (2\zeta_{-\alpha,-\beta} + 1)(\tfrac{1}{2}(\gamma_0^{(\alpha,\beta)}\gamma_0^{(-\alpha,-\beta)})^{\frac{1}{2}} + 1).$$

If, in addition, $-1 < \alpha, \beta \leqslant 0$ *or* $0 < \alpha, \beta < 1$, *then for* $0 \leqslant \mu \leqslant 1$,

$$||\widetilde{P}^{1,0}_{N,\alpha,\beta}v - v||_{\mu,\chi^{(\alpha,\beta)}} \leqslant c\widetilde{C}^{3,\mu}_{\alpha,\beta}(N(N+\alpha+\beta))^{\frac{\mu-r}{2}}|v|_{r,\chi^{(\alpha,\beta)},*}, \quad (3.23)$$

where $\widetilde{C}^{3,\mu}_{\alpha,\beta} = (\widetilde{C}^{3,1}_{\alpha,\beta})^{\mu}(\widetilde{C}^{3,0}_{\alpha,\beta})^{1-\mu}$ *and*

$$\widetilde{C}^{3,0}_{\alpha,\beta} = \begin{cases} q_{\alpha,\beta}(2\zeta_{-\alpha,-\beta}+1)(\frac{1}{2}(\gamma_0^{(\alpha,\beta)}\gamma_0^{(-\alpha,-\beta)})^{\frac{1}{2}}+1)^2 & \textit{for } -1<\alpha,\beta\leqslant 0,\\ q^{(1)}_{\alpha,\beta}(\frac{1}{2}(\gamma_0^{(\alpha,\beta)}\gamma_0^{(-\alpha,-\beta)})^{\frac{1}{2}}+1)^2 & \textit{for } 0<\alpha,\beta<1. \end{cases}$$

Proof. Let

$$\phi^*(x) = \int_{-1}^{x} P_{N-1,\alpha,\beta}\partial_y v(y)\,dy, \quad \phi(x) = \phi^*(x) - \tfrac{1}{2}\phi^*(1)(1+x).$$

Clearly $\phi \in \mathcal{P}^0_N$. By projection theorem and an argument as in derivation of (3.19), we have that

$$|\widetilde{P}^{1,0}_{N,\alpha,\beta}v - v|_{1,\chi^{(\alpha,\beta)}} \leqslant |v-\phi|_{1,\chi^{(\alpha,\beta)}} \leqslant ||P_{N-1,\alpha,\beta}\partial_x v - \partial_x v||_{\chi^{(\alpha,\beta)}} + \tfrac{1}{2}(\gamma_0^{(\alpha,\beta)})^{\frac{1}{2}}|\phi^*(1)|$$
$$\leqslant (\tfrac{1}{2}(\gamma_0^{(\alpha,\beta)}\gamma_0^{(-\alpha,-\beta)})^{\frac{1}{2}}+1)||P_{N-1,\alpha,\beta}\partial_x v - \partial_x v||_{\chi^{(\alpha,\beta)}}. \tag{3.24}$$

The above with (2.15) and Theorem 2.1 leads to (3.22).

We now turn to prove (3.23) with $\mu = 0$. Let $g \in L^2_{\chi^{(\alpha,\beta)}}(\Lambda)$ and consider an auxiliary problem. It is to seek $w \in H^1_{0,\chi^{(\alpha,\beta)}}(\Lambda)$ such that

$$\widetilde{a}_{\alpha,\beta}(w,z) = (g,z)_{\chi^{(\alpha,\beta)}} \quad \forall z \in H^1_{0,\chi^{(\alpha,\beta)}}(\Lambda). \tag{3.25}$$

In sense of distributions,

$$-\partial_x(\chi^{(\alpha,\beta)}(x)\partial_x w(x)) = g(x)\chi^{(\alpha,\beta)}(x). \tag{3.26}$$

Let $u(x) = \chi^{(\alpha,\beta)}(x)\partial_x w(x)$. We take $z = w$ in (3.25). Due to (2.15), we assert that $||u||_{\chi^{(-\alpha,-\beta)}} = |w|_{1,\chi^{(\alpha,\beta)}} \leqslant 2\zeta_{-\alpha,-\beta}||g||_{\chi^{(\alpha,\beta)}}$. Moreover, by taking the $L^2_{\chi^{(-\alpha,-\beta)}}$-norms of both sides of (3.26), we obtain that

$$|u|_{1,\chi^{(-\alpha,-\beta)}} = ||g(x)\chi^{(\alpha,\beta)}(x)||_{\chi^{(-\alpha,-\beta)}} = ||g||_{\chi^{(\alpha,\beta)}}. \tag{3.27}$$

Next, let $\mathcal{T}_{N,\alpha,\beta}$ be the same as in (2.12), and

$$u_N(x) = \mathcal{T}_{N,\alpha,\beta}u(x) - \frac{1}{2}\chi^{(\alpha,\beta)}(x)\int_\Lambda \mathcal{T}_{N,\alpha,\beta}u(x)\chi^{(-\alpha,-\beta)}(x)\,dx, \tag{3.28}$$

$$w_N(x) = \int_{-1}^{x} u_N(y)\chi^{(-\alpha,-\beta)}(y)\,dy.$$

Clearly $u_N \in \mathcal{U}_{N,\alpha,\beta)}(\Lambda)$ and $w_N \in \mathcal{P}^0_N$. Since $w \in H^1_{0,\chi^{(\alpha,\beta)}}(\Lambda)$ and $u(x)\chi^{(-\alpha,-\beta)}(x) = \partial_x w(x)$, we have

$$\int_\Lambda \mathcal{T}_{N,\alpha,\beta}u(x)\chi^{(-\alpha,-\beta)}(x)\,dx = \int_\Lambda (\mathcal{T}_{N,\alpha,\beta}u(x) - u(x))\chi^{(-\alpha,-\beta)}(x)\,dx.$$

For simplicity, we denote the right side of the above formula by η. If $-1<\alpha,\beta\leqslant 0$, then by (2.15), (3.27), (3.28) and Lemma 2.1,

$$\begin{aligned}|w-w_N|_{1,\chi^{(\alpha,\beta)}} &= ||u-u_N||_{\chi^{(-\alpha,-\beta)}} \leqslant ||\mathcal{T}_{N,\alpha,\beta}u-u||_{\chi^{(-\alpha,-\beta)}} + \tfrac{1}{2}(\gamma_0^{(\alpha,\beta)})^{\frac{1}{2}}|\eta| \\ &\leqslant (\tfrac{1}{2}(\gamma_0^{(\alpha,\beta)}\gamma_0^{(-\alpha,-\beta)})^{\frac{1}{2}}+1)||\mathcal{T}_{N,\alpha,\beta}u-u||_{\chi^{(-\alpha,-\beta)}} \\ &\leqslant cq_{\alpha,\beta}(\tfrac{1}{2}(\gamma_0^{(\alpha,\beta)}\gamma_0^{(-\alpha,-\beta)})^{\frac{1}{2}}+1)(N(N+\alpha+\beta))^{-\frac{1}{2}}||u||_{1,\chi^{(-\alpha,-\beta)}} \\ &\leqslant cq_{\alpha,\beta}(2\zeta_{-\alpha,-\beta}+1)(\tfrac{1}{2}(\gamma_0^{(\alpha,\beta)}\gamma_0^{(-\alpha,-\beta)})^{\frac{1}{2}}+1) \\ &\quad\times (N(N+\alpha+\beta))^{-\frac{1}{2}}||g||_{\chi^{(\alpha,\beta)}}. \end{aligned} \tag{3.29}$$

If $0<\alpha,\beta<1$, then $u(x)=\chi^{(\alpha,\beta)}(x)\partial_x w(x)\to 0$ as $|x|\to 1$. So we can use (2.17) to estimate $||\mathcal{T}_{N,\alpha,\beta}u-u||_{\chi^{(-\alpha,-\beta)}}$. By the same procedure as in derivation of (3.29), we assert that

$$|w-w_N|_{1,\chi^{(\alpha,\beta)}} \leqslant cq_{\alpha,\beta}^{(1)}(\tfrac{1}{2}(\gamma_0^{(\alpha,\beta)}\gamma_0^{(-\alpha,-\beta)})^{\frac{1}{2}}+1)(N(N+\alpha+\beta))^{-\frac{1}{2}}||g||_{\chi^{(\alpha,\beta)}}. \tag{3.30}$$

Taking $z=\widetilde{P}^{1,0}_{N,\alpha,\beta}v-v$ in (3.25), and using (3.21), (3.24), (3.29) and (3.30), we deduce that

$$\begin{aligned}|(\widetilde{P}^{1,0}_{N,\alpha,\beta}v-v,g)_{\chi^{(\alpha,\beta)}}| &= |\widetilde{a}_{\alpha,\beta}(\widetilde{P}^{1,0}_{N,\alpha,\beta}v-v,w-w_N)| \\ &\leqslant c\widetilde{C}^{3,0}_{\alpha,\beta}(N(N+\alpha+\beta))^{-\frac{r}{2}}||g||_{\chi^{(\alpha,\beta)}}|v|_{r,\chi^{(\alpha,\beta)},*}.\end{aligned}$$

Consequently,

$$||\widetilde{P}^{1,0}_{N,\alpha,\beta}v-v||_{\chi^{(\alpha,\beta)}} \leqslant cd\widetilde{C}^{3,0}_{\alpha,\beta}(N(N+\alpha+\beta))^{-\frac{r}{2}}|v|_{r,\chi^{(\alpha,\beta)},*}. \tag{3.31}$$

Finally, we obtain the desired result for $0<\mu<1$ by space interpolation. □

When we apply Jacobi approximations to non-singular problems, we should use another orthogonal projection. To do this, let

$$\bar{a}_{\alpha,\beta}(u,v)=(\partial_x u,\partial_x(\chi^{(\alpha,\beta)}v)).$$

Lemma 3.1. *If* $-1<\alpha,\beta<1$, *then for any* $u,v\in H^1_{0,\chi^{(\alpha,\beta)}}(\Lambda)$,

$$|\bar{a}_{\alpha,\beta}(u,v)|\leqslant M_{\alpha,\beta}|u|_{1,\chi^{(\alpha,\beta)}}|v|_{1,\chi^{(\alpha,\beta)}}, \quad \bar{a}_{\alpha,\beta}(v,v)\geqslant L^{-1}_{\alpha,\beta}|v|^2_{1,\chi^{(\alpha,\beta)}}. \tag{3.32}$$

where $M_{\alpha,\beta} = 4\zeta_{-\alpha,-\beta} + 1$ *and*

$$L_{\alpha,\beta} = \begin{cases} 1 & \textit{for } 0 \leqslant \alpha, \beta \leqslant 1, \\ \sqrt{2}(16\zeta^2_{\alpha,\beta}(\alpha^2+\beta^2)+1)^{\frac{1}{2}} & \textit{for } -1 < \alpha, \beta < 0, \\ \sqrt{2}(16\alpha^2\zeta^2_{\alpha,-\beta}+1)^{\frac{1}{2}} & \textit{for } -1 < \alpha \leqslant 0 \leqslant \beta < 1, \\ \sqrt{2}(16\beta^2\zeta^2_{-\alpha,\beta}+1)^{\frac{1}{2}} & \textit{for } -1 < \beta \leqslant 0 \leqslant \alpha < 1. \end{cases}$$

Proof. Since $-1 < \alpha, \beta < 1$, we have from (2.15) that

$$\begin{aligned} |\bar{a}_{\alpha,\beta}(u,v)| &\leqslant |(\partial_x u, \partial_x v)_{\chi^{(\alpha,\beta)}} + (\partial_x u, v\partial_x \chi^{(\alpha,\beta)})| \\ &\leqslant |u|_{1,\chi^{(\alpha,\beta)}}|v|_{1,\chi^{(\alpha,\beta)}} + 2|u|_{1,\chi^{(\alpha,\beta)}}||v||_{\chi^{(\alpha-2,\beta-2)}} \leqslant M_{\alpha,\beta}|u|_{1,\chi^{(\alpha,\beta)}}|v|_{1,\chi^{(\alpha,\beta)}}. \end{aligned}$$

We now prove the second result of (3.32). A direct calculation gives that

$$\bar{a}_{\alpha,\beta}(v,v) = |v|^2_{1,\chi^{(\alpha,\beta)}} + \frac{1}{2}(u^2, W_{\alpha,\beta})_{\chi^{(\alpha-2,\beta-2)}},$$

$$W_{\alpha,\beta}(x) = (\alpha+\beta)(1-\alpha-\beta)x^2 + 2(\alpha-\beta)(1-\alpha-\beta)x + \alpha + \beta - (\alpha-\beta)^2.$$

It can be checked that $W_{\alpha,\beta}(x) \geqslant 0$, provided that

$$\begin{aligned} &(\alpha+\beta)(\alpha+\beta-1) \geqslant 0, \quad W_{\alpha,\beta}(-1) = -4\beta^2 + 4\beta \geqslant 0, \\ &\quad W_{\alpha,\beta}(1) = -4\alpha^2 + 4\alpha \geqslant 0 \end{aligned} \tag{3.33}$$

or

$$\begin{aligned} &(\alpha+\beta)(\alpha+\beta-1) \leqslant 0, \\ &\quad 4(\alpha-\beta)^2(\alpha+\beta-1)^2 + 4(\alpha+\beta)(\alpha+\beta-1)(\alpha+\beta-(\alpha-\beta)^2) \leqslant 0. \end{aligned} \tag{3.34}$$

If $0 \leqslant \alpha, \beta \leqslant 1$, then both (3.33) and (3.34) are valid. This fact implies the second result of (3.32) with $0 \leqslant \alpha, \beta \leqslant 1$.

Next, let $-1 < \alpha, \beta < 0$ and $u(x) = \chi^{(\alpha,\beta)}(x)v(x)$. We know from Lemma 3.8 of [20] that $u \in H^1_{0,\chi^{(-\alpha,-\beta)}}(\Lambda)$. So by the previous result,

$$\bar{a}_{\alpha,\beta}(v,v) = \bar{a}_{-\alpha,-\beta}(u,u) \geqslant |u|^2_{1,\chi^{(-\alpha,-\beta)}}. \tag{3.35}$$

On the other hand, by (2.15),

$$\begin{aligned} |v|^2_{1,\chi^{(\alpha,\beta)}} &\leqslant 2|u|^2_{1,\chi^{(-\alpha,-\beta)}} + 8(\alpha^2+\beta^2)||u||^2_{\chi^{(\alpha-2,\beta-2)}} \\ &\leqslant 2(16\zeta^2_{\alpha,\beta}(\alpha^2+\beta^2)+1)|u|^2_{1,\chi^{(-\alpha,-\beta)}}. \end{aligned} \tag{3.36}$$

A combination of (3.35) and (3.36) leads to the second result of (3.32).

Thirdly, let $-1 < \alpha \leqslant 0 \leqslant \beta < 1$ and $u(x) = (1-x)^{\alpha}v(x)$. By Lemma 3.8 of [20], $u \in H^1_{0,\chi^{(-\alpha,0)}}(\Lambda)$. Using (2.15) again gives that

$$\begin{aligned} |v|^2_{1,\chi^{(\alpha,\beta)}} &= |(1-x)^{-\alpha}u|^2_{1,\chi^{(\alpha,\beta)}} \leqslant 2|u|^2_{1,\chi^{(-\alpha,\beta)}} + 2\alpha^2||u||^2_{\chi^{(-\alpha-2,\beta)}} \\ &\leqslant 2|u|^2_{1,\chi^{(-\alpha,\beta)}} + 8\alpha^2||u||^2_{\chi^{(-\alpha-2,\beta-2)}} \leqslant 2(16\alpha^2\zeta^2_{\alpha,-\beta}+1)|u|^2_{1,\chi^{(-\alpha,\beta)}}. \end{aligned} \tag{3.37}$$

Moreover, due to $-1<\alpha\leqslant 0\leqslant\beta<1$, we have that

$$|u|^2_{1,\chi^{(-\alpha,\beta)}}\leqslant |u|^2_{1,\chi^{(-\alpha,\beta)}}-2\alpha(\alpha+1)||u||^2_{\chi^{(-\alpha-2,\beta)}}+2\beta(1-\beta)||u||^2_{\chi^{(-\alpha,\beta-2)}}$$
$$=(\partial_x((1-x)^{-\alpha}u),\partial_x((1+x)^{\beta}u))=\bar{a}_{\alpha,\beta}(v,v). \tag{3.38}$$

A combination of (3.37) and (3.38) leads to the second result of (3.32).

We can deal with the case $-1<\beta\leqslant 0\leqslant\alpha<1$ in the same manner. $\square$

The orthogonal projection $\bar{P}^{1,0}_{N,\alpha,\beta}:H^1_{0,\chi^{(\alpha,\beta)}}(\Lambda)\to\mathcal{P}^0_N$ is defined by

$$\bar{a}_{\alpha,\beta}(\bar{P}^{1,0}_{N,\alpha,\beta}v-v,\phi)=0\quad\forall\phi\in\mathcal{P}^0_N. \tag{3.39}$$

Theorem 3.5. *If $-1<\alpha,\beta<1$, then for any $v\in H^1_{0,\chi^{(\alpha,\beta)}}(\Lambda)\cap H^r_{\chi^{(\alpha,\beta)},*}(\Lambda), r\in\mathbb{N}, r\geqslant 1$ and $0\leqslant\mu\leqslant 1$,*

$$||\bar{P}^{1,0}_{N,\alpha,\beta}v-v||_{\mu,\chi^{(\alpha,\beta)}}\leqslant c\bar{C}^{3,\mu}_{\alpha,\beta}(N(N+\alpha+\beta))^{\frac{\mu-r}{2}}|v|_{r,\chi^{(\alpha,\beta)},*} \tag{3.40}$$

where $\bar{C}^{3,\mu}_{\alpha,\beta}=(\bar{C}^{3,1}_{\alpha,\beta})^{\mu}(\bar{C}^{3,0}_{\alpha,\beta})^{1-\mu}$ and

$$\bar{C}^{3,0}_{\alpha,\beta}=L_{\alpha,\beta}M^2_{\alpha,\beta}(\tfrac{1}{2}(\gamma_0^{(\alpha,\beta)}\gamma_0^{(-\alpha,-\beta)})^{\frac{1}{2}}+1)^2,$$
$$\bar{C}^{3,1}_{\alpha,\beta}=(2\zeta_{-\alpha,-\beta}+1)L^{\frac{1}{2}}_{\alpha,\beta}M^{\frac{1}{2}}_{\alpha,\beta}(\tfrac{1}{2}(\gamma_0^{(\alpha,\beta)}\gamma_0^{(-\alpha,-\beta)})^{\frac{1}{2}}+1).$$

Proof. By projection theorem, Lemma 3.1, (3.24) and Theorem 2.1, we have that,

$$|\bar{P}^{1,0}_{N,\alpha,\beta}v-v|^2_{1,\chi^{(\alpha,\beta)}}\leqslant|\widetilde{P}^{1,0}_{N,\alpha,\beta}v-v|^2_{1,\chi^{(\alpha,\beta)}}\leqslant L_{\alpha,\beta}\bar{a}_{\alpha,\beta}(\widetilde{P}^{1,0}_{N,\alpha,\beta}v-v,\widetilde{P}^{1,0}_{N,\alpha,\beta}v-v)$$
$$\leqslant L_{\alpha,\beta}M_{\alpha,\beta}(\tfrac{1}{2}(\gamma_0^{(\alpha,\beta)}\gamma_0^{(-\alpha,-\beta)})^{\frac{1}{2}}+1)^2||P_{N-1,\alpha,\beta}\partial_x v-\partial_x v||^2_{\chi^{(\alpha,\beta)}}$$
$$\leqslant cL_{\alpha,\beta}M_{\alpha,\beta}(\tfrac{1}{2}(\gamma_0^{(\alpha,\beta)}\gamma_0^{(-\alpha,-\beta)})^{\frac{1}{2}}+1)^2$$
$$\times(N(N+\alpha+\beta))^{1-r}|v|^2_{r,\chi^{(\alpha,\beta)},*}. \tag{3.41}$$

The above with (2.15) implies

$$||\bar{P}^{1,0}_{N,\alpha,\beta}v-v||_{1,\chi^{(\alpha,\beta)}}\leqslant c\bar{C}^{3,1}_{\alpha,\beta}(N(N+\alpha+\beta))^{\frac{1-r}{2}}|v|_{r,\chi^{(\alpha,\beta)},*}. \tag{3.42}$$

Now, let $g\in L^2_{\chi^{(\alpha-1,\beta-1)}}(\Lambda)$ and consider the auxiliary problem

$$\bar{a}_{\alpha,\beta}(w,z)=(g,z)_{\chi^{(\alpha-1,\beta-1)}}\quad\forall z\in H^1_{0,\chi^{(\alpha,\beta)}}(\Lambda). \tag{3.43}$$

By (2.15),

$$|(g,z)_{\chi^{(\alpha-1,\beta-1)}}|\leqslant||g||_{\chi^{(\alpha-1,\beta-1)}}||z||_{\chi^{(\alpha-2,\beta-2)}}\leqslant 2\zeta_{-\alpha,-\beta}||g||_{\chi^{(\alpha-1,\beta-1)}}|z|_{1,\chi^{(\alpha,\beta)}}.$$

Thus (3.43) has a unique solution in $H^1_{0,\chi^{(\alpha,\beta)}}(\Lambda)$. Moreover, in sense of distributions, $\partial^2_x w(x)=-(1-x^2)^{-1}g(x)$. Accordingly $||\partial^2_x w(1-x^2)^{\frac{1}{2}}||_{\chi^{(\alpha,\beta)}}=||g||_{\chi^{(\alpha-1,\beta-1)}}$. Taking

$z = \bar{P}^{1,0}_{N,\alpha,\beta}v - v$ in (3.43), we use Lemma 3.1 and (3.41) to reach that

$$\begin{aligned}|(g, \bar{P}^{1,0}_{N,\alpha,\beta}v - v)_{\chi^{(\alpha-1,\beta-1)}}| &= |\bar{a}_{\alpha,\beta}(w - \bar{P}^{1,0}_{N,\alpha,\beta}w, \bar{P}^{1,0}_{N,\alpha,\beta}v - v)| \\ &\leqslant M_{\alpha,\beta}|\bar{P}^{1,0}_{N,\alpha,\beta}w - w|_{1,\chi^{(\alpha,\beta)}}|\bar{P}^{1,0}_{N,\alpha,\beta}v - v|_{1,\chi^{(\alpha,\beta)}} \\ &\leqslant cL_{\alpha,\beta}M^2_{\alpha,\beta}(\tfrac{1}{2}(\gamma_0^{(\alpha,\beta)}\gamma_0^{(-\alpha,-\beta)})^{\frac{1}{2}} + 1)^2 \\ &\quad \times (N(N+\alpha+\beta))^{-\frac{r}{2}}|v|_{r,\chi^{(\alpha,\beta)},*}||g||_{\chi^{(\alpha-1,\beta-1)}}.\end{aligned}$$

Therefore,

$$||\bar{P}^{1,0}_{N,\alpha,\beta}v - v||_{\chi^{(\alpha,\beta)}} \leqslant ||\bar{P}^{1,0}_{N,\alpha,\beta}v - v||_{\chi^{(\alpha-1,\beta-1)}} \leqslant c\bar{C}^{3,0}_{\alpha,\beta}(N(N+\alpha+\beta))^{-\frac{r}{2}}|v|_{r,\chi^{(\alpha,\beta)},*}.$$

From the above, (3.42) and space interpolation, the desired result (3.40) follows. □

In the previous parts, we studied Jacobi approximations with the parameters $\alpha, \beta, \gamma, \delta > -1$. But in some practical problems, we also need to consider certain critical cases, in which some parameters are equal to -1, see [5,14,15]. Here, we consider the case with $\alpha = 1, \beta = 0, \gamma = -1$ and $\delta = 0$. The notations $H^1_{0,1,0-1,0}(\Lambda), P^{1,0}_{N,1,0-1,0}$ and $a_{1,0-1,0}(\cdot,\cdot)$ have the same meanings as in Theorem 3.3.

Theorem 3.6. *For any* $v \in H^1_{0,1,0-1,0}(\Lambda) \cap H^r_{\chi^{(\bar{\alpha},0)},*}(\Lambda), r \in \mathbb{N}, r \geqslant 1$ *and* $\bar{\alpha} \in (-1,1)$,

$$||P^{1,0}_{N,1,0-1,0}v - v||_{1,1,0-1,0} \leqslant cC^{4,1}_{\bar{\alpha}}((N(N+\bar{\alpha}))^{\frac{1-r}{2}}|v|_{r,\chi^{(\bar{\alpha},0)},*}, \tag{3.44}$$

where

$$C^{4,1}_{\bar{\alpha}} = 2^{\frac{1-\bar{\alpha}}{2}}(2\zeta_{-\bar{\alpha},0} + 1)^{\frac{1}{2}}(\tfrac{1}{2}(\gamma_0^{(\bar{\alpha},0)}\gamma_0^{(-\bar{\alpha},0)})^{\frac{1}{2}} + 1).$$

If, in addition, $\bar{\alpha} \in [0,1)$, *then for* $0 \leqslant \mu \leqslant 1$,

$$||P^{1,0}_{N,1,0-1,0}v - v||_{\mu,\chi^{(1,0)}} \leqslant c(\Delta_{\bar{\alpha},1})^{1-\mu}(C^{4,1}_{\bar{\alpha}})^{2-\mu}(N(N+\bar{\alpha}))^{\frac{\mu-r}{2}}|v|_{r,\chi^{(\bar{\alpha},0)},*}. \tag{3.45}$$

Proof. We first prove (3.44). Let $\widetilde{P}^{1,0}_{N,\bar{\alpha},0}$ be the orthogonal projection as in (3.21). Then by projection theorem, (2.15), (3.24) and Theorem 3.4,

$$\begin{aligned}||P^{1,0}_{N,1,0-1,0}v - v||^2_{1,1,0-1,0} &\leqslant |\widetilde{P}^{1,0}_{N,\bar{\alpha},0}v - v|^2_{1,\chi^{(1,0)}} + ||\widetilde{P}^{1,0}_{N,\bar{\alpha},0}v - v||^2_{\chi^{(-1,0)}} \\ &\leqslant 2^{1-\bar{\alpha}}(2\zeta_{-\bar{\alpha},0} + 1)|\widetilde{P}^{1,0}_{N,\bar{\alpha},0}v - v|^2_{1,\chi^{(\bar{\alpha},0)}} \\ &\leqslant c2^{1-\bar{\alpha}}(2\zeta_{-\bar{\alpha},0} + 1)(\tfrac{1}{2}(\gamma_0^{(\bar{\alpha},0)}\gamma_0^{(-\bar{\alpha},0)})^{\frac{1}{2}} + 1)^2 \\ &\quad \times (N(N+\bar{\alpha}))^{1-r}|v|^2_{r,\chi^{(\bar{\alpha},0)},*}.\end{aligned} \tag{3.46}$$

We next prove (3.45). By (3.44),

$$\begin{aligned}||P^{1,0}_{N,1,0-1,0}v - v||_{1,\chi^{(1,0)}} &\leqslant 2||P^{1,0}_{N,1,0-1,0}v - v||_{1,1,0,-1,0} \\ &\leqslant cC^{4,1}_{\bar{\alpha}}(N(N+\bar{\alpha}))^{\frac{1-r}{2}}|v|_{r,\chi^{(\bar{\alpha},0)},*}.\end{aligned} \tag{3.47}$$

Now, let $g \in L^2_{\chi^{(1,0)}}(\Lambda)$ and consider the auxiliary problem

$$a_{1,0-1,0}(w,z) = (g,z)_{\chi^{(1,0)}} \quad \forall z \in H^1_{0,\chi^{(1,0)}}(\Lambda). \tag{3.48}$$

Following the same lines as in derivation of (IV.3.19) of [5], we can verify that $||w||_{2,\chi^{(1,0)}} \leqslant c||g||_{\chi^{(1,0)}}$. Since $0 \leqslant \bar{\alpha} < 1$, we get from (3.44) that,

$$\begin{aligned}||P^{1,0}_{N,1,0-1,0}w - w||_{1,1,0-1,0} &\leqslant cC^{4,1}_{\bar{\alpha}}(N(N+\bar{\alpha}))^{-\frac{1}{2}}||\partial_x^2 w||_{\chi^{(\bar{\alpha}+1,1)}} \\ &\leqslant cC^{4,1}_{\bar{\alpha}}(N(N+\bar{\alpha}))^{-\frac{1}{2}}\Delta_{\bar{\alpha},1}||w||_{2,\chi^{(1,0)}} \\ &\leqslant c\Delta_{\bar{\alpha},1}C^{4,1}_{\bar{\alpha}}(N(N+\bar{\alpha}))^{-\frac{1}{2}}||g||_{\chi^{(1,0)}}.\end{aligned} \tag{3.49}$$

Taking $z = P^{1,0}_{N,1,0-1,0}v - v$ in (3.48), we use (3.44), (3.48) and (3.49) to obtain that

$$|(P^{1,0}_{N,1,0-1,0}v - v, g)_{\chi^{(1,0)}}| \leqslant c\Delta_{\bar{\alpha},1}(C^{4,1}_{\bar{\alpha}})^2(N(N+\bar{\alpha}))^{-\frac{r}{2}}||g||_{\chi^{(1,0)}}|v|_{r,\chi^{(\bar{\alpha},0)},*}.$$

Consequently,

$$||P^{1,0}_{N,1,0-1,0}v - v||_{\chi^{(1,0)}} \leqslant c\Delta_{\bar{\alpha},1}(C^{4,1}_{\bar{\alpha}})^2(N(N+\bar{\alpha}))^{-\frac{r}{2}}|v|_{r,\chi^{(\bar{\alpha},0)},*}. \tag{3.50}$$

Finally, we obtain (3.45) with $0 < \mu < 1$ by using (3.47), (3.50) and space interpolation. □

4. Jacobi–Gauss-type interpolations

In this section, we study Jacobi–Gauss-type interpolations. Let $\zeta^{(\alpha,\beta)}_{G,N,j}, \zeta^{(\alpha,\beta)}_{R,N,j}$ and $\zeta^{(\alpha,\beta)}_{L,N,j}$, $0 \leqslant j \leqslant N$, be the zeros of polynomials $J^{(\alpha,\beta)}_{N+1}(x), (1+x)J^{(\alpha,\beta+1)}_N(x)$ and $(1-x^2)\partial_x J^{(\alpha,\beta)}_N(x)$, respectively. They are arranged in decreasing order. Denote by $\omega^{(\alpha,\beta)}_{Z,N,j}$, $0 \leqslant j \leqslant N$, $Z = G, R, L$, the corresponding Christoffel numbers such that,

$$\int_\Lambda \phi(x)\chi^{(\alpha,\beta)}(x)\,dx = \sum_{j=0}^{N} \phi(\zeta^{(\alpha,\beta)}_{Z,N,j})\omega^{(\alpha,\beta)}_{Z,N,j} \quad \forall \phi \in \mathcal{P}_{2N+\lambda_Z}, \tag{4.1}$$

where $\lambda_Z = 1, 0$ and -1 for $Z = G, R$ and L, respectively. There hold the relations (see [20]),

$$\zeta^{(\alpha,\beta)}_{R,N,j} = \zeta^{(\alpha,\beta+1)}_{G,N-1,j}, \quad \omega^{(\alpha,\beta)}_{R,N,j} = (1+\zeta^{(\alpha,\beta+1)}_{G,N-1,j})^{-1}\omega^{(\alpha,\beta+1)}_{G,N-1,j}, \quad 0 \leqslant j \leqslant N-1, \tag{4.2}$$

$$\begin{aligned}&\zeta^{(\alpha,\beta)}_{L,N,j} = \zeta^{(\alpha+1,\beta+1)}_{G,N-2,j-1}, \quad \omega^{(\alpha,\beta)}_{L,N,j} = (1-(\zeta^{(\alpha+1,\beta+1)}_{G,N-2,j-1})^2)^{-1}\omega^{(\alpha+1,\beta+1)}_{G,N-2,j-1}, \\ &\quad 1 \leqslant j \leqslant N-1.\end{aligned} \tag{4.3}$$

Let z_N and w_N be two sequences and $w_N \neq 0$. If $\frac{z_N}{w_N} \to 1$ as $N \to \infty$, then we write $z_N \cong w_N$. According to (15.3.10) of [27], (4.2) and (4.3), we have that

$$\omega_{G,N,j}^{(\alpha,\beta)} \cong \frac{\pi}{N+1}(1-\zeta_{G,N,j}^{(\alpha,\beta)})^{\alpha+\frac{1}{2}}(1+\zeta_{G,N,j}^{(\alpha,\beta)})^{\beta+\frac{1}{2}}, \quad 0 \leqslant j \leqslant N, \tag{4.4}$$

$$\omega_{R,N,j}^{(\alpha,\beta)} \cong \frac{\pi}{N}(1-\zeta_{R,N,j}^{(\alpha,\beta)})^{\alpha+\frac{1}{2}}(1+\zeta_{R,N,j}^{(\alpha,\beta)})^{\beta+\frac{1}{2}}, \quad 0 \leqslant j \leqslant N-1, \tag{4.5}$$

$$\omega_{L,N,j}^{(\alpha,\beta)} \cong \frac{\pi}{N-1}(1-\zeta_{L,N,j}^{(\alpha,\beta)})^{\alpha+\frac{1}{2}}(1+\zeta_{L,N,j}^{(\alpha,\beta)})^{\beta+\frac{1}{2}}, \quad 1 \leqslant j \leqslant N-1. \tag{4.6}$$

We now define the discrete inner product and norm by

$$(u,v)_{\chi^{(\alpha,\beta)},Z,N} = \sum_{j=0}^{N} u(\zeta_{Z,N,j}^{(\alpha,\beta)})v(\zeta_{Z,N,j}^{(\alpha,\beta)})\omega_{Z,N,j}^{(\alpha,\beta)}, \quad ||v||_{\chi^{(\alpha,\beta)},Z,N} = (v,v)^{\frac{1}{2}}_{\chi^{(\alpha,\beta)},Z,N}.$$

By (4.1), we have

$$(\phi,\psi)_{\chi^{(\alpha,\beta)},Z,N} = (\phi,\psi)_{\chi^{(\alpha,\beta)}} \quad \forall \phi \cdot \psi \in \mathcal{P}_{2N+\lambda_Z}, \quad Z = G, R, L. \tag{4.7}$$

For any $\phi \in \mathcal{P}_N$ (see [20]),

$$||\phi||_{\chi^{(\alpha,\beta)}} \leqslant ||\phi||_{\chi^{(\alpha,\beta)},L,N} \leqslant (2+(\alpha+\beta+1)N^{-1})^{\frac{1}{2}}||\phi||_{\chi^{(\alpha,\beta)}}. \tag{4.8}$$

Let $\Lambda_{Z,N}^{(\alpha,\beta)} = \{x \mid x = \zeta_{Z,N,j}^{(\alpha,\beta)}, 0 \leqslant j \leqslant N\}$. The Jacobi–Gauss-type interpolant $\mathcal{I}_{Z,N,\alpha,\beta}v \in \mathcal{P}_N$ such that

$$\mathcal{I}_{Z,N,\alpha,\beta}v(x) = v(x), \quad x \in \Lambda_{Z,N}^{(\alpha,\beta)}, \tag{4.9}$$

where for $Z = G, R$ and L. They are named as Jacobi–Gauss, Jacobi–Gauss–Radau and Jacobi–Gauss–Lobatto interpolation, respectively.

We shall estimate the difference between $\mathcal{I}_{Z,N,\alpha,\beta}v$ and v in non-uniformly Jacobi-weighted Sobolev spaces. In the sequel,

$$x = \cos\theta, \quad \theta \in [0,\pi], \quad \theta_{Z,N,j}^{(\alpha,\beta)} = \arccos \zeta_{Z,N,j}^{(\alpha,\beta)}, \quad Z = G, R, L, \quad 0 \leqslant j \leqslant N. \tag{4.10}$$

We first present a result on distribution of Jacobi–Gauss interpolation nodes.

Lemma 4.1. *Let* $\widetilde{N} = N + \frac{1}{2}(\alpha+\beta) + \frac{3}{2}$. *We have that*

(i) *if* $-\frac{1}{2} < \alpha, \beta < \frac{1}{2}$, *then*

$$(j + \tfrac{1}{2}(\alpha+\beta+1))\pi\widetilde{N}^{-1} < \theta_{G,N,j}^{(\alpha,\beta)} < (j+1)\pi\widetilde{N}^{-1}, \quad 0 \leqslant j \leqslant N, \tag{4.11}$$

(ii) *if* $\alpha > 0$ *and* $\beta > -1$, *then*

$$(j + \tfrac{1}{2}\alpha)\pi\widetilde{N}^{-1} < \theta_{G,N,j}^{(\alpha,\beta)} < (j + \tfrac{1}{2}\alpha + 1)\pi\widetilde{N}^{-1}, \quad 0 \leqslant j \leqslant N, \tag{4.12}$$

(iii) *if* $\alpha > -1$ *and* $\beta > 0$, *then*

$$(j + \tfrac{1}{2}(\alpha+1))\pi\widetilde{N}^{-1} < \theta_{G,N,j}^{(\alpha,\beta)} < (j + \tfrac{1}{2}(\alpha+3))\pi\widetilde{N}^{-1}, \quad 0 \leqslant j \leqslant N, \tag{4.13}$$

(iv) *if*

$$-1<\alpha\leqslant-\tfrac{1}{2},\quad -1<\beta\leqslant 0,\quad or\ -\tfrac{1}{2}<\alpha\leqslant 0,\quad -1<\beta\leqslant-\tfrac{1}{2}, \tag{4.14}$$

then

$$\theta^{(\alpha,\beta)}_{G,N,j}=(j\pi+O(1))(N+1)^{-1},\quad 0\leqslant j\leqslant N. \tag{4.15}$$

Proof. The results (4.11) and (4.15) come from (6.3.7) and Theorem 8.9.1 of [27]. We now prove (4.12). By Theorem 8.21.8 of [27],

$$J^{(\alpha,\beta)}_{N+1}(\cos\theta)=(N+1)^{-\frac{1}{2}}F(\theta)\cos(\widetilde{N}\theta+\gamma)+O((N+1)^{-\frac{3}{2}}), \tag{4.16}$$

where

$$F(\theta)=\pi^{-\frac{1}{2}}\left(\sin\frac{\theta}{2}\right)^{-\alpha-\frac{1}{2}}\left(\cos\frac{\theta}{2}\right)^{-\beta-\frac{1}{2}},\quad \gamma=-\tfrac{1}{4}(2\alpha+1)\pi. \tag{4.17}$$

Let $\widetilde{\theta}^{(\alpha,\beta)}_{N,j}=((j-\frac{1}{4})\pi-\gamma)\widetilde{N}^{-1}, 0\leqslant j\leqslant N+1$. Since $\alpha>0$ and $\beta>-1$, we have $\widetilde{\theta}^{(\alpha,\beta)}_{N,j}\in(0,\pi), 0\leqslant j\leqslant N+1$. Moreover, $-\alpha-\frac{1}{2}<0, -\beta-1<0$ and $\sin x\geqslant\frac{2}{\pi}x$ for $x\in[0,\frac{\pi}{2}]$. Thus we have from (4.17) that for $0\leqslant j\leqslant N$,

$$\begin{aligned}F(\widetilde{\theta}^{(\alpha,\beta)}_{N,j})&=\frac{1}{\sqrt{\pi}}\left(\sin\frac{\widetilde{\theta}^{(\alpha,\beta)}_{N,j}}{2}\right)^{-\alpha-\frac{1}{2}}\left(\cos\frac{\widetilde{\theta}^{(\alpha,\beta)}_{N,j}}{2}\right)^{-\beta-\frac{1}{2}}\\&\geqslant\frac{1}{\sqrt{\pi}}\left(\cos\frac{\widetilde{\theta}^{(\alpha,\beta)}_{N,j}}{2}\right)^{\frac{1}{2}}=\frac{1}{\sqrt{\pi}}\left(\sin\frac{\pi-\widetilde{\theta}^{(\alpha,\beta)}_{N,j}}{2}\right)^{\frac{1}{2}}\geqslant\sqrt{\frac{N-j+\frac{\beta}{2}+\frac{3}{2}}{\pi\widetilde{N}}}.\end{aligned} \tag{4.18}$$

Meanwhile we have from (4.16) that

$$\begin{aligned}J^{(\alpha,\beta)}_{N+1}(\cos\widetilde{\theta}^{(\alpha,\beta)}_{N,j})&=(N+1)^{-\frac{1}{2}}F(\widetilde{\theta}^{(\alpha,\beta)}_{N,j})\cos(j\pi+\tfrac{\alpha\pi}{2})+O((N+1)^{-\frac{3}{2}})\\&=(-1)^j(N+1)^{-\frac{1}{2}}F(\widetilde{\theta}^{(\alpha,\beta)}_{N,j})\cos(\tfrac{\alpha\pi}{2})+O((N+1)^{-\frac{3}{2}}),\\&\quad 0\leqslant j\leqslant N+1.\end{aligned}$$

The above with (4.18) implies $\mathrm{sgn}(J^{(\alpha,\beta)}_{N+1}(\cos\widetilde{\theta}^{(\alpha,\beta)}_{N,j}))=(-1)^j$. Therefore there exists at least one zero of $J^{(\alpha,\beta)}_{N+1}(\cos\theta)$ in each subinterval $(\widetilde{\theta}^{(\alpha,\beta)}_{N,j},\widetilde{\theta}^{(\alpha,\beta)}_{N,j+1})$. Since the zeros of $J^{(\alpha,\beta)}_{N+1}(\cos\theta)$ are distinct, there is exactly one zero in each subinterval $(\widetilde{\theta}^{(\alpha,\beta)}_{N,j},\widetilde{\theta}^{(\alpha,\beta)}_{N,j+1}), 0\leqslant j\leqslant N$. This leads to (4.12).

We next prove (4.13). Let $\widehat{\theta}^{(\alpha,\beta)}_{N,j}=((j+\frac{1}{4})\pi-\gamma)\widetilde{N}^{-1}, 0\leqslant j\leqslant N+1$. Since $\alpha>-1$ and $\beta>0$, we have $\widehat{\theta}^{(\alpha,\beta)}_{N,j}\in(0,\pi)$. Thanks to $-\alpha-1<0$ and $-\beta-\frac{1}{2}<0$, we obtain that

$$F(\widehat{\theta}^{(\alpha,\beta)}_{N,j})\geqslant\frac{1}{\sqrt{\pi}}\left(\sin\frac{\hat{\theta}^{(\alpha,\beta)}_{N,j}}{2}\right)^{\frac{1}{2}}\geqslant\sqrt{\frac{j+\frac{\alpha}{2}+\frac{1}{2}}{\pi\widetilde{N}}},\quad 0\leqslant j\leqslant N+1.$$

By this fact and (4.16), we assert that $\operatorname{sgn}(J^{(\alpha,\beta)}_{N+1}(\cos\widehat{\theta}^{(\alpha,\beta)}_{N,j})) = (-1)^{j+1}$. The rest part of proof is clear. $\square$

The results (4.12) and (4.13) refine the results of Theorem 8.9.1 of [27] in the three special cases. But they cover nearly the whole range $\alpha,\beta > -1$. How to improve the result of Theorem 8.9.1 of [27] with $\alpha \leqslant -\frac{1}{2}, \beta \leqslant 0$ and $\alpha \leqslant 0, \beta < -\frac{1}{2}$ is still an open problem.

We next present a result on the stability of Jacobi–Gauss interpolation. For simplicity, we denote by $d_1 \geqslant 1$ a constant such that $d_1 \to 1$ as $N \to \infty$, and set

$$Q^{k,l}_{\alpha,\beta} = \max((2\alpha+2k+1)^2, (2\beta+2l+1)^2).$$

Theorem 4.1. *For any* $v \in H^1_{\chi^{(\alpha+k,\beta+l)},\Lambda}(\Lambda)$, $k,l \in \mathbb{N}$ *and* $0 \leqslant k+l \leqslant 1$,

$$\begin{aligned}||\mathcal{I}_{G,N,\alpha,\beta}v||_{\chi^{(\alpha+k,\beta+l)}} \leqslant{}& A^{(\alpha,\beta)}_{1,N}||v||_{\chi^{(\alpha+k,\beta+l)}} \\ &+ d_2\left(N\left(N+\tfrac{\alpha}{2}+\tfrac{\beta}{2}+1\right)\right)^{-\frac{1}{2}}||\partial_x v||_{\chi^{(\alpha+k+1,\beta+l+1)}},\end{aligned} \tag{4.19}$$

where

$$A^{(\alpha,\beta)}_{1,N} = \begin{cases} \sqrt{d_1\pi}\left(\dfrac{4}{(1-\alpha-\beta)\pi} + \dfrac{\pi Q^{k,l}_{\alpha,\beta}}{(1+\alpha+\beta)^2}\right)^{\frac{1}{2}} & \\ \quad\times\left(1+\dfrac{\alpha+\beta+1}{2(N+1)}\right)^{\frac{1}{2}} & \text{if } -\frac{1}{2}<\alpha,\beta<\frac{1}{2}, \\ \sqrt{d_1\pi}\left(\dfrac{2}{\pi} + \pi Q^{k,l}_{\alpha,\beta}\max(\alpha^{-2},(1+\beta)^{-2})\right)^{\frac{1}{2}} & \\ \quad\times\left(1+\dfrac{\alpha+\beta+1}{2(N+1)}\right)^{\frac{1}{2}} & \text{if } \alpha>0 \text{ and } \beta>-1, \\ \sqrt{d_1\pi}\left(\dfrac{2}{\pi} + \pi Q^{k,l}_{\alpha,\beta}\max((1+\alpha)^{-2},\beta^{-2})\right)^{\frac{1}{2}} & \\ \quad\times\left(1+\dfrac{\alpha+\beta+1}{2(N+1)}\right)^{\frac{1}{2}} & \text{if } \alpha>-1 \text{ and } \beta>0, \\ c(\alpha,\beta) & \text{if (4.14) holds,} \end{cases}$$

while $d_2 = 2\pi\sqrt{d_1}$, *in the first three cases and* $d_2 = c(\alpha,\beta)$ *if* (4.14) *holds. Hereafter* $c(\alpha,\beta)$ *is a positive constant depending only on* α *and* β.

Proof. Let $0 < a_0 < \frac{\pi}{2} < a_2 < \pi$, and $\{I_j\}_{j=0}^N$ be a set of subintervals such that

$$[a_0, a_2] = \bigcup_{j=0}^N \bar{I}_j, \quad \theta_{G,N,j}^{(\alpha,\beta)} \in I_j, \quad I_i \cap I_j = \emptyset \quad \forall i \neq j.$$

We denote by $|I_j|$ the length of subinterval I_j, and set $\hat{v}(\theta) = v(\cos\theta)$. It is noted that for any $v \in H^1(a,b)$,

$$\max_{x\in[a,b]} |v(x)| \leqslant (b-a)^{-\frac{1}{2}} ||v||_{L^2(a,b)} + (b-a)^{\frac{1}{2}} ||\partial_x v||_{L^2(a,b)}.$$

Obviously, $(\mathcal{I}_{G,N,\alpha,\beta} v(x))^2 \chi^{(k,l)}(x) \in \mathcal{P}_{2N+1}$. Thus by (4.4), (4.7) and (4.10),

$$\begin{aligned}
&||\mathcal{I}_{G,N,\alpha,\beta} v||^2_{\chi^{(\alpha+k,\beta+l)}} \\
&\quad = ||\mathcal{I}_{G,N,\alpha,\beta} v \chi^{(\frac{k}{2},\frac{l}{2})}||^2_{\chi^{(\alpha,\beta)},G,N} = \sum_{j=0}^N v^2(\zeta_{G,N,j}^{(\alpha,\beta)}) \chi^{(k,l)}(\zeta_{G,N,j}^{(\alpha,\beta)}) \omega_{G,N,j}^{(\alpha,\beta)} \\
&\quad \cong \frac{2^{\alpha+\beta+k+l+1}\pi}{N+1} \sum_{j=0}^N \hat{v}^2(\theta_{G,N,j}^{(\alpha,\beta)}) \left(\sin\frac{\theta_{G,N,j}^{(\alpha,\beta)}}{2}\right)^{2\alpha+2k+1} \left(\cos\frac{\theta_{G,N,j}^{(\alpha,\beta)}}{2}\right)^{2\beta+2l+1} \\
&\quad \leqslant \frac{2^{\alpha+\beta+k+l+1} d_1 \pi}{N+1} \sum_{j=0}^N \sup_{\theta\in I_j} \left| \hat{v}(\theta) \left(\sin\frac{\theta}{2}\right)^{\alpha+k+\frac{1}{2}} \left(\cos\frac{\theta}{2}\right)^{\beta+l+\frac{1}{2}} \right|^2 \\
&\quad \leqslant \frac{2^{\alpha+\beta+k+l+1} d_1 \pi}{N+1} \sum_{j=0}^N \left(\frac{2}{|I_j|} ||\hat{v}(\theta) \left(\sin\frac{\theta}{2}\right)^{\alpha+k+\frac{1}{2}} \left(\cos\frac{\theta}{2}\right)^{\beta+l+\frac{1}{2}} ||^2_{L^2(I_j)} \right. \\
&\qquad \left. + 2|I_j| \cdot ||\partial_\theta (\hat{v}(\theta) \left(\sin\frac{\theta}{2}\right)^{\alpha+k+\frac{1}{2}} \left(\cos\frac{\theta}{2}\right)^{\beta+l+\frac{1}{2}})||^2_{L^2(I_j)} \right) \\
&\quad \leqslant \frac{2^{\alpha+\beta+k+l+1} d_1 \pi}{N+1} \left(\max_{0\leqslant j\leqslant N} \frac{2}{|I_j|} ||\hat{v}(\theta) \left(\sin\frac{\theta}{2}\right)^{\alpha+k+\frac{1}{2}} \left(\cos\frac{\theta}{2}\right)^{\beta+l+\frac{1}{2}} ||^2_{L^2(0,\pi)} \right. \\
&\qquad \left. + 2 \max_{0\leqslant j\leqslant N} |I_j| \cdot ||\partial_\theta \left(\hat{v}(\theta) \left(\sin\frac{\theta}{2}\right)^{\alpha+k+\frac{1}{2}} \left(\cos\frac{\theta}{2}\right)^{\beta+l+\frac{1}{2}} \right) ||^2_{L^2(a_0,a_2)} \right).
\end{aligned} \tag{4.20}$$

A direct calculation gives that

$$\partial_\theta \left(\left(\sin\frac{\theta}{2}\right)^{\alpha+k+\frac{1}{2}} \left(\cos\frac{\theta}{2}\right)^{\beta+l+\frac{1}{2}} \right) = D(\theta) \left(\sin\frac{\theta}{2}\right)^{\alpha+k-\frac{1}{2}} \left(\cos\frac{\theta}{2}\right)^{\beta+l-\frac{1}{2}}, \tag{4.21}$$

where

$$D(\theta) = \tfrac{1}{4}(\alpha - \beta + k - l + (\alpha+\beta+k+l+1)\cos\theta).$$

It can be checked that $\max_{\theta\in[0,\pi]} D^2(\theta)\leqslant\frac{1}{16}Q^{k,l}_{\alpha,\beta}$. Therefore by (4.21),

$$\begin{aligned}
&||\partial_\theta\left(\hat{v}(\theta)\left(\sin\frac{\theta}{2}\right)^{\alpha+k+\frac12}\left(\cos\frac{\theta}{2}\right)^{\beta+l+\frac12}\right)||^2_{L^2(a_0,a_2)}\\
&\quad\leqslant 2||\partial_\theta\hat{v}(\theta)\left(\sin\frac{\theta}{2}\right)^{\alpha+k+\frac12}\left(\cos\frac{\theta}{2}\right)^{\beta+l+\frac12}||^2_{L^2(0,\pi)}\\
&\qquad+\frac12 Q^{k,l}_{\alpha,\beta}\max_{\theta\in[a_0,a_2]}\frac{1}{\sin^2\theta}\cdot||\hat{v}(\theta)\left(\sin\frac{\theta}{2}\right)^{\alpha+k+\frac12}\left(\cos\frac{\theta}{2}\right)^{\beta+l+\frac12}||^2_{L^2(0,\pi)}. \qquad (4.22)
\end{aligned}$$

It is easy to show that

$$\begin{aligned}
&\frac{d\theta}{dx}=-\frac{1}{\sqrt{1-x^2}},\quad \partial_\theta\hat{v}(\theta)=-\sqrt{1-x^2}\,\partial_x v(x),\\
&\max_{\theta\in[a_0,a_1]}\frac{1}{\sin^2\theta}\leqslant\frac{\pi^2}{4}\max\left(\frac{1}{a_0^2},\frac{1}{(\pi-a_1)^2}\right).
\end{aligned}$$

So a combination of (4.20) and (4.22) leads to that

$$||\mathcal{I}_{G,N,\alpha,\beta}v||^2_{\chi^{(\alpha+k,\beta+l)}}\leqslant p^{(\alpha,\beta)}_{1,N}||v||^2_{\chi^{(\alpha+k,\beta+l)}}+p^{(\alpha,\beta)}_{2,N}||\partial_x v||^2_{\chi^{(\alpha+k+1,\beta+l+1)}}, \qquad (4.23)$$

where

$$\begin{aligned}
p^{(\alpha,\beta)}_{1,N}&=\frac{d_1\pi}{N+1}\left(2\max_{0\leqslant j\leqslant N}\frac{1}{|I_j|}+\frac{\pi^2}{4}Q^{k,l}_{\alpha,\beta}\max_{0\leqslant j\leqslant N}|I_j|\max\left(\frac{1}{a_0^2},\frac{1}{(\pi-a_2)^2}\right)\right),\\
p^{(\alpha,\beta)}_{2,N}&=\frac{4d_1\pi}{N+1}\max_{0\leqslant j\leqslant N}|I_j|.
\end{aligned}$$

If $-\frac12<\alpha,\beta<\frac12$, then by (4.11), we can take

$$\begin{aligned}
&I_j=((j+\tfrac12(\alpha+\beta+1))\pi\widetilde{N}^{-1},(j+1)\pi\widetilde{N}^{-1}),\quad |I_j|=\tfrac12(1-\alpha-\beta)\pi\widetilde{N}^{-1},\\
&\qquad 0\leqslant j\leqslant N.
\end{aligned}$$

In this case, $a_0=\frac12(\alpha+\beta+1)\pi\widetilde{N}^{-1}$ and $a_1=\pi(N+1)\widetilde{N}^{-1}=\pi-a_0$. Thus

$$\begin{aligned}
p^{(\alpha,\beta)}_{1,N}&=d_1\pi\frac{\widetilde{N}}{N+1}\left(\frac{4}{(1-\alpha-\beta)\pi}+Q^{k,l}_{\alpha,\beta}\frac{(1-\alpha-\beta)\pi}{2(1+\alpha+\beta)^2}\right),\\
p^{(\alpha,\beta)}_{2,N}&=\frac{2(1-\alpha-\beta)d_1\pi^2}{\widetilde{N}(N+1)}.
\end{aligned}$$

This with (4.23) leads to the desired result for $-\frac12<\alpha,\beta<\frac12$.

Next, let $\alpha>0$ and $\beta>-1$. According to (4.12), we take

$$I_j=\left(\left(j+\frac{\alpha}{2}\right)\pi\widetilde{N}^{-1},\left(j+\frac{\alpha}{2}+1\right)\pi\widetilde{N}^{-1}\right),\quad |I_j|=\pi\widetilde{N}^{-1},\quad 0\leqslant j\leqslant N.$$

In this case, $a_0 = \frac{1}{2}\alpha\pi\widetilde{N}^{-1}$ and $a_1 = (N + \frac{1}{2}\alpha + 1)\pi\widetilde{N}^{-1}$. So we have that

$$p_{1,N}^{(\alpha,\beta)} = d_1\pi\left(\frac{2}{\pi} + \pi Q_{\alpha,\beta}^{k,l}\max(\alpha^{-2},(1+\beta)^{-2})\right)\left(1 + \frac{1+\alpha+\beta}{2(N+1)}\right),$$
$$p_{2,N}^{(\alpha,\beta)} = \frac{4d_1\pi^2}{\widetilde{N}(N+1)}.$$

This with (4.23) leads to the desired result. We can derive the desired result for $\alpha > -1$ and $\beta > 0$, similarly.

Finally, if (4.14) holds, then we know from (4.15) that $\theta_{G,N,j}^{(\alpha,\beta)} \in I_j$, I_j being of size $c(N+1)^{-1}$. Let $a_0 = O((N+1)^{-1})$ and $a_1 = (N\pi + O(1))(N+1)^{-1}$. By an argument as in the previous parts, we have that

$$||\mathcal{I}_{G,N,\alpha,\beta}v||_{\chi^{(\alpha+k,\beta+l)}} \leqslant c(\alpha,\beta)(||v||_{\chi^{(\alpha+k,\beta+l)}} + (N+1)^{-1}||\partial_x v||_{\chi^{(\alpha+k+1,\beta+l+1)}}). \qquad \square$$

We are now in position of presenting the main results on the Jacobi–Gauss interpolation. We first estimate the difference between $\mathcal{I}_{G,N,\alpha,\beta}v$ and v in the space $H^r_{\chi^{(\alpha,\beta)},A}(\Lambda)$.

Theorem 4.2. *Let $k,l,r \in \mathbb{N}$ and $0 \leqslant k+l \leqslant 1$. For any $v \in H^r_{\chi^{(\alpha+k,\beta+l)},A}(\Lambda)$, $r \geqslant 1$ and $0 \leqslant \mu \leqslant r$,*

$$||\mathcal{I}_{G,N,\alpha,\beta}v - v||_{\mu,\chi^{(\alpha+k,\beta+l)},A} \leqslant cB_{1,N}^{(\alpha,\beta)}(N(N+\alpha+\beta+k+l))^{\frac{\mu-r}{2}}|v|_{r,\chi^{(\alpha+k,\beta+l)},A}, \quad (4.24)$$

where $B_{1,N}^{(\alpha,\beta)} = A_{1,N}^{(\alpha,\beta)} + d_2 + 1$.

Proof. By using Lemma 2.5, Theorem 2.1 and Theorem 4.1, we obtain that for $\mu \in \mathbb{N}$,

$$\begin{aligned}
&||\partial_x^\mu(\mathcal{I}_{G,N,\alpha,\beta}v - P_{N,\alpha+k,\beta+l}v)||_{\chi^{(\alpha+k+\mu,\beta+l+\mu)}}\\
&\quad\leqslant (N(N+\alpha+\beta+k+l+2\mu))^{\frac{\mu}{2}}||\mathcal{I}_{G,N,\alpha,\beta}(P_{N,\alpha+k,\beta+l}v - v)||_{\chi^{(\alpha+k,\beta+l)}}\\
&\quad\leqslant (N(N+\alpha+\beta+k+l+2\mu))^{\frac{\mu}{2}}\Bigg(A_{1,N}^{(\alpha,\beta)}||P_{N,\alpha+k,\beta+l}v - v||_{\chi^{(\alpha+k,\beta+l)}}\\
&\qquad + d_2\left(N\left(N + \frac{\alpha}{2} + \frac{\beta}{2} + 1\right)\right)^{-\frac{1}{2}}||\partial_x(P_{N,\alpha+k,\beta+l}v - v)||_{\chi^{(\alpha+k+1,\beta+l+1)}}\Bigg)\\
&\quad\leqslant c(A_{1,N}^{(\alpha,\beta)} + d_2)(N(N+\alpha+\beta+k+l))^{\frac{\mu-r}{2}}|v|_{r,\chi^{(\alpha+k,\beta+l)},A}.
\end{aligned}$$

Then the result (4.24) with $\mu \in \mathbb{N}$ follows from the above estimate, Theorem 2.1 and a triangle inequality. Finally, we use (2.7) to obtain the desired result. $\square$

It is more important to estimate the error of Jacobi–Gauss interpolation in the space $H^\mu_{\alpha,\beta,\gamma,\delta}(\Lambda)$. In this case, we can choose Jacobi–Gauss interpolation associated with the weighted function $\chi^{(\alpha,\beta)}(x)$ or $\chi^{(\gamma,\delta)}(x)$. We state the results in the following two theorems.

Theorem 4.3. *If* (2.31) *and* (2.43) *hold, then for any* $v \in H^r_{\chi^{(\alpha,\beta)},*}(\Lambda), r \in \mathbb{N}$ *and* $r \geqslant 1$,

$$|\mathcal{I}_{G,N,\alpha,\beta}v - v|_{1,\chi^{(\alpha,\beta)}} \leqslant cG^{(1)}_{N,\alpha,\beta,\gamma,\delta}(N(N+\alpha+\beta))^{1-\frac{r}{2}}|v|_{r,\chi^{(\alpha,\beta)},*}, \tag{4.25}$$

$$||\mathcal{I}_{G,N,\alpha,\beta}v - v||_{\chi^{(\gamma,\delta)}} \leqslant c\kappa_{\gamma,\delta}\Delta_{\gamma-\alpha+2,\delta-\beta+2}G^{(1)}_{N,\alpha,\beta,\gamma,\delta}(N(N+\alpha+\beta))^{1-\frac{r}{2}}|v|_{r,\chi^{(\alpha,\beta)},*}, \tag{4.26}$$

where

$$G^{(1)}_{N,\alpha,\beta,\gamma,\delta} = (\alpha-\beta+1)(B^{(\alpha,\beta)}_{1,N} + C^{1,0}_{\alpha,\beta,\alpha,\beta}) + (N(N+\alpha+\beta))^{-\frac{1}{2}}.$$

If, in addition, $\alpha \leqslant \gamma$, $\beta \leqslant \delta$, *then*

$$||\mathcal{I}_{G,N,\alpha,\beta}v - v||_{\chi^{(\gamma,\delta)}} \leqslant c\Delta_{\gamma-\alpha,\delta-\beta}B^{(\alpha,\beta)}_{1,N}(N(N+\alpha+\beta))^{-\frac{r}{2}}|v|_{r,\chi^{(\alpha,\beta)},A}. \tag{4.27}$$

Proof. By Lemma 2.6, Theorem 4.2 and an argument as in derivation of (3.5),

$$\begin{aligned}|\mathcal{I}_{G,N,\alpha,\beta}v - v|_{1,\chi^{(\alpha,\beta)}} &\leqslant |\mathcal{I}_{G,N,\alpha,\beta}v - P^1_{N,\alpha,\beta,\alpha,\beta}v|_{1,\chi^{(\alpha,\beta)}} + |P^1_{N,\alpha,\beta,\alpha,\beta}v - v|_{1,\chi^{(\alpha,\beta)}}\\ &\leqslant c(\alpha-\beta+1)N(N+\alpha+\beta)(||\mathcal{I}_{G,N,\alpha,\beta}v - v||_{\chi^{(\alpha,\beta)}}\\ &\quad + ||P^1_{N,\alpha,\beta,\alpha,\beta}v - v||_{\chi^{(\alpha,\beta)}}) + |P^1_{N,\alpha,\beta,\alpha,\beta}v - v|_{1,\chi^{(\alpha,\beta)}}\\ &\leqslant cG^{(1)}_{N,\alpha,\beta,\gamma,\delta}(N(N+\alpha+\beta))^{1-\frac{r}{2}}|v|_{r,\chi^{(\alpha,\beta)},*}.\end{aligned}$$

Since $\mathcal{I}_{G,N,\alpha,\beta}v(\zeta^{(\alpha,\beta)}_{G,N,j}) = v(\zeta^{(\alpha,\beta)}_{G,N,j}), 0 \leqslant j \leqslant N$, we can choose $x_0 = \zeta^{(\alpha,\beta)}_{G,N,k}$ such that $|x_0| = \min_{0\leqslant j\leqslant N}|\zeta^{(\alpha,\beta)}_{G,N,j}|$. Then by Lemma 2.3 with (2.31),

$$||\mathcal{I}_{G,N,\alpha,\beta}v - v||_{\chi^{(\gamma,\delta)}} \leqslant c\kappa_{\gamma,\delta}\Delta_{\gamma-\alpha+2,\delta-\beta+2}|\mathcal{I}_{G,N,\alpha,\beta}v - v|_{1,\chi^{(\alpha,\beta)}}.$$

The above two estimates imply (4.26).

If $\alpha \leqslant \gamma$ and $\beta \leqslant \delta$, then (4.27) follows from Theorem 4.2 and the fact that

$$||\mathcal{I}_{G,N,\alpha,\beta}v - v||_{\chi^{(\gamma,\delta)}} \leqslant \Delta_{\gamma-\alpha,\delta-\beta}||\mathcal{I}_{G,N,\alpha,\beta}v - v||_{\chi^{(\alpha,\beta)}}. \qquad \square$$

Theorem 4.4. *If* (2.43) *holds and*

$$\gamma \leqslant \alpha \leqslant \gamma+1, \quad \delta \leqslant \beta \leqslant \delta+1, \tag{4.28}$$

then for any $v \in H^r_{\chi^{(\alpha,\beta)},*}(\Lambda), r \in \mathbb{N}$ *and* $r \geqslant 1$,

$$|\mathcal{I}_{G,N,\gamma,\delta}v - v|_{1,\chi^{(\alpha,\beta)}} \leqslant cG^{(2)}_{\alpha,\beta,\gamma,\delta}(N(N+\alpha+\beta))^{1-\frac{r}{2}}|v|_{r,\chi^{(\alpha,\beta)},*}, \tag{4.29}$$

$$||\mathcal{I}_{G,N,\gamma,\delta}v - v||_{\chi^{(\gamma,\delta)}} \leqslant cB^{(\gamma,\delta)}_{1,N}(N(N+\alpha+\beta))^{-\frac{r}{2}}|v|_{r,\chi^{(\gamma+r,\delta+r)},A}, \tag{4.30}$$

where

$$G^{(2)}_{N,\alpha,\beta,\gamma,\delta}=(\alpha-\beta+1)\Delta_{\alpha-\gamma,\beta-\delta}(B^{(\gamma,\delta)}_{1,N}\Delta_{\gamma-\alpha+1,\delta-\beta+1}+C^{1,0}_{\alpha,\beta,\gamma,\delta})\\ +\ (N(N+\alpha+\beta))^{-\frac{1}{2}}.$$

Proof. Due to (4.28), we have from Theorem 4.2 that

$$\begin{aligned}||\mathcal{I}_{G,N,\gamma,\delta}v-v||_{\chi^{(\gamma,\delta)}}&\leqslant cB^{(\gamma,\delta)}_{1,N}(N(N+\alpha+\beta))^{-\frac{r}{2}}|v|_{r,\chi^{(\gamma+r,\delta+r)},A}\\&\leqslant cB^{(\gamma,\delta)}_{1,N}\Delta_{\gamma-\alpha+1,\delta-\beta+1}(N(N+\alpha+\beta))^{-\frac{r}{2}}|v|_{r,\chi^{(\alpha,\beta)},*}.\end{aligned}$$

Next, we use (4.28), Theorem 3.1 and Lemma 2.6 to obtain that

$$\begin{aligned}|\mathcal{I}_{G,N,\gamma,\delta}v-P^1_{N,\alpha,\beta,\gamma,\delta}v|_{1,\chi^{(\alpha,\beta)}}&\leqslant c(\alpha-\beta+1)N(N+\alpha+\beta)||\mathcal{I}_{G,N,\gamma,\delta}v\\&\quad-\ P^1_{N,\alpha,\beta,\gamma,\delta}v||_{\chi^{(\alpha,\beta)}}\\&\leqslant c(\alpha-\beta+1)\Delta_{\alpha-\gamma,\beta-\delta}N(N+\alpha+\beta)(||\mathcal{I}_{G,N,\gamma,\delta}v\\&\quad-v||_{\chi^{(\gamma,\delta)}}+||P^1_{N,\alpha,\beta,\gamma,\delta}v-v||_{\chi^{(\gamma,\delta)}})\\&\leqslant c(\alpha-\beta+1)\Delta_{\alpha-\gamma,\beta-\delta}(B^{(\gamma,\delta)}_{1,N}\Delta_{\gamma-\alpha+1,\delta-\beta+1}\\&\quad+\ C^{1,0}_{\alpha,\beta,\gamma,\delta})(N(N+\alpha+\beta))^{1-\frac{r}{2}}|v|_{r,\chi^{(\alpha,\beta)},*}.\end{aligned}$$

Finally, the result (4.29) follows from the above estimate and an argument as in derivation of (3.5). □

We now turn to the Jacobi–Gauss–Radau interpolation. To shorten the paper, we only present the results which can be proved in the same manner as in the proof of the last four theorems. But we should use Lemma 2.2 in the proof of Theorem 4.6, and use Theorem 3.4 in the proof of Theorem 4.10, respectively.

Theorem 4.5. *Let* $k,l\in\mathbb{N},0\leqslant k\leqslant l\leqslant 1$ *and* $l<\beta+1$. *For any* $v\in H^1_{\chi^{(\alpha+k,\beta-l)},A}(\Lambda)$ *with* $v(-1)=0$,

$$||\mathcal{I}_{R,N,\alpha,\beta}v||_{\chi^{(\alpha+k,\beta-l)}}\leqslant A^{(\alpha,\beta)}_{2,N}||v||_{\chi^{(\alpha+k,\beta-l)}}\\ +2\pi\sqrt{d_1}\left(N\left(N+\frac{\alpha}{2}+\frac{\beta}{2}\right)\right)^{-\frac{1}{2}}||\partial_x v||_{\chi^{(\alpha+k+1,\beta-l+1)}},\tag{4.31}$$

where d_1 *is the same as in Theorem* 4.1, *and*

$$A^{(\alpha,\beta)}_{2,N}=\sqrt{d_2\pi}\left(\frac{2}{\pi}+\pi Q^{k,-l}_{\alpha,\beta}\max((1+\alpha)^{-2},(1+\beta)^{-2})\right)^{\frac{1}{2}}\left(1+\frac{2+\alpha+\beta}{2N}\right)^{\frac{1}{2}}.$$

Theorem 4.6. *Let $k, l, r \in \mathbb{N}, 0 \leqslant k \leqslant l \leqslant 1$ and $l < \beta + 1$. For any $v \in H^r_{\chi^{(\alpha+k,\beta-l)},A}(\Lambda), r \geqslant 1$ and $0 \leqslant \mu \leqslant r$,*

$$||\mathcal{I}_{R,N,\alpha,\beta}v - v||_{\mu,\chi^{(\alpha+k,\beta-l)},A} \leqslant cB^{(\alpha,\beta)}_{2,N}(N(N+\alpha+\beta+k-l))^{\frac{\mu-r}{2}}|v|_{r,\chi^{(\alpha+k,\beta-l)},A}, \quad (4.32)$$

where $B^{(\alpha,\beta)}_{2,N} = \sigma_{\alpha,\beta}(A^{(\alpha,\beta)}_{2,N} + d_1^{\frac{1}{2}} + 1)$.

Theorem 4.7. *If* (2.31) *and* (2.43) *hold, then for any $v \in H^r_{\chi^{(\alpha,\beta)},*}(\Lambda), r \in \mathbb{N}$ and $r \geqslant 1$,*

$$|\mathcal{I}_{R,N,\alpha,\beta}v - v|_{1,\chi^{(\alpha,\beta)}} \leqslant cR^{(1)}_{N,\alpha,\beta,\gamma,\delta}(N(N+\alpha+\beta))^{1-\frac{r}{2}}|v|_{r,\chi^{(\alpha,\beta)},*}, \quad (4.33)$$

$$||\mathcal{I}_{R,N,\alpha,\beta}v - v||_{\chi^{(\gamma,\delta)}} \leqslant c\kappa_{\gamma,\delta}\Delta_{\gamma-\alpha+2,\delta-\beta+2}R^{(1)}_{N,\alpha,\beta,\gamma,\delta}(N(N+\alpha+\beta))^{1-\frac{r}{2}}|v|_{r,\chi^{(\alpha,\beta)},*}, \quad (4.34)$$

where

$$R^{(1)}_{N,\alpha,\beta,\gamma,\delta} = (\alpha-\beta+1)(B^{(\alpha,\beta)}_{2,N} + C^{1,0}_{\alpha,\beta,\alpha,\beta}) + (N(N+\alpha+\beta))^{-\frac{1}{2}}.$$

If, in addition, $\alpha \leqslant \gamma$, $\beta \leqslant \delta$, then

$$||\mathcal{I}_{R,N,\alpha,\beta}v - v||_{\chi^{(\gamma,\delta)}} \leqslant c\Delta_{\gamma-\alpha,\delta-\beta}B^{(\alpha,\beta)}_{2,N}(N(N+\alpha+\beta))^{-\frac{r}{2}}|v|_{r,\chi^{(\alpha,\beta)},A}. \quad (4.35)$$

Theorem 4.8. *If* (2.43) *and* (4.28) *hold, then for any $v \in H^r_{\chi^{(\alpha,\beta)},*}(\Lambda), r \in \mathbb{N}$ and $r \geqslant 1$,*

$$|\mathcal{I}_{R,N,\gamma,\delta}v - v|_{1,\chi^{(\alpha,\beta)}} \leqslant cR^{(2)}_{\alpha,\beta,\gamma,\delta}(N(N+\alpha+\beta))^{1-\frac{r}{2}}|v|_{r,\chi^{(\alpha,\beta)},*}, \quad (4.36)$$

$$||\mathcal{I}_{R,N,\gamma,\delta}v - v||_{\chi^{(\gamma,\delta)}} \leqslant cB^{(\gamma,\delta)}_{2,N}(N(N+\alpha+\beta))^{-\frac{r}{2}}|v|_{r,\chi^{(\gamma+r,\delta+r)},A}, \quad (4.37)$$

where

$$R^{(2)}_{N,\alpha,\beta,\gamma,\delta} = (\alpha-\beta+1)\Delta_{\alpha-\gamma,\beta-\delta}(B^{(\gamma,\delta)}_{2,N}\Delta_{\gamma-\alpha+1,\delta-\beta+1} + C^{1,0}_{\alpha,\beta,\gamma,\delta}) + (N(N+\alpha+\beta))^{-\frac{1}{2}}.$$

Finally, we present the main results on the Jacobi–Gauss–Lobatto interpolation.

Theorem 4.9. *Let $k, l \in \mathbb{N}, 0 \leqslant k, l \leqslant 1, k < \alpha + 1$ and $l < \beta + 1$. For any $v \in H^1_{\chi^{(\alpha-k,\beta-l)},A}(\Lambda)$ with $v(\pm 1) = 0$,*

$$||\mathcal{I}_{L,N,\alpha,\beta}v||_{\chi^{(\alpha-k,\beta-l)}} \leqslant A^{(\alpha,\beta)}_{3,N}||v||_{\chi^{(\alpha-k,\beta-l)}} + 2\pi\sqrt{d_1}\left(N\left(N+\frac{\alpha}{2}+\frac{\beta}{2}\right)\right)^{-\frac{1}{2}}||\partial_x v||_{\chi^{(\alpha-k+1,\beta-l+1)}}, \quad (4.38)$$

where d_1 is the same as in Theorem 4.1, *and*

$$A^{(\alpha,\beta)}_{3,N} = \sqrt{d_1\pi}\left(\frac{2}{\pi} + \pi Q^{-k,-l}_{\alpha,\beta}\max((1+\alpha)^{-2},(2+\beta)^{-2})\right)^{\frac{1}{2}}\left(1+\frac{\alpha+\beta+3}{2(N-1)}\right)^{\frac{1}{2}}.$$

Theorem 4.10. *If* $-1<\alpha,\beta\leqslant 0$ *or* $0<\alpha,\beta<1$, *then for any* $v\in H^r_{\chi^{(\alpha,\beta)},*}(\Lambda), r\in\mathbb{N}$ *and* $r\geqslant 1$,

$$||\mathcal{I}_{L,N,\alpha,\beta}v-v||_{\chi^{(\alpha,\beta)}}\leqslant cL^{(1)}_{N,\alpha,\beta}(N(N+\alpha+\beta))^{-\frac{r}{2}}|v|_{r,\chi^{(\alpha,\beta)},*}, \tag{4.39}$$

where

$$L^{(1)}_{N,\alpha,\beta}=((\gamma_0^{(-\alpha,-\beta)}\gamma_0^{(\alpha,\beta)})^{\frac{1}{2}}+1)\left(\widetilde{C}^{3,0}_{\alpha,\beta}(A^{(\alpha,\beta)}_{3,N}+1)+2\pi\sqrt{d_1}\widetilde{C}^{3,1}_{\alpha,\beta}\right).$$

If, in addition, (2.43) *holds, then we have*

$$|\mathcal{I}_{L,N,\alpha,\beta}v-v|_{1,\chi^{(\alpha,\beta)}}\leqslant cL^{(2)}_{N,\alpha,\beta}(N(N+\alpha+\beta))^{1-\frac{r}{2}}|v|_{r,\chi^{(\alpha,\beta)},*},$$

where

$$\begin{aligned}L^{(2)}_{N,\alpha,\beta}&=((\gamma_0^{(-\alpha,-\beta)}\gamma_0^{(\alpha,\beta)})^{\frac{1}{2}}+1)((\alpha-\beta+1)\widetilde{C}^{3,0}_{\alpha,\beta}(N(N+\alpha+\beta))^{-1}\\&\quad+\widetilde{C}^{3,1}_{(\alpha,\beta)}(N(N+\alpha+\beta))^{-\frac{1}{2}})+(\alpha-\beta+1)L^{(1)}_{N,\alpha,\beta}.\end{aligned} \tag{4.40}$$

Let $c_{\alpha,\beta,\gamma,\delta}$ be a generic positive constant depending only on $\alpha,\beta,\gamma,\delta$. If (2.43) is not fulfilled, then the right sides of (4.25), (4.26), (4.29), (4.33), (4.34), (4.36) and (4.40) become $c_{\alpha,\beta,\gamma,\delta}N^{2-r}|v|_{r,\chi^{(\alpha,\beta)},*}$, while the right sides of (4.27) and (4.35) become $c_{\alpha,\beta,\gamma,\delta}N^{-r}|v|_{r,\chi^{(\alpha,\beta)},A}$. Similarly, the right sides of (4.30) and (4.37) now turn to be $c_{\alpha,\beta,\gamma,\delta}N^{-r}|v|_{r,\chi^{(\gamma+r,\delta+r)},A}$. In fact, in these cases, $|v|_{r,\chi^{(\gamma+r,\delta+r)},A}\leqslant\Delta_{\gamma-\alpha+1,\delta-\beta+1}|v|_{r,\chi^{(\alpha,\beta)},*}$.

5. Concluding discussions

As we know, Babuška and Guo [3] studied symmetric Jacobi approximations in Jacobi-weighted Sobolev spaces, in which the weight for $\partial_x^r v$ is $(1-x^2)^{\alpha+r}$. Meanwhile Bernardi and Maday [6] considered symmetric Jacobi approximations in Sobolev spaces with the uniform weight $(1-x^2)^{\alpha}$. In this paper, we established a series of results on general Jacobi approximations and Jacobi–Gauss-type interpolations. They generalize the results of [3,6] and so could be used for numerical solutions of various problems, such as

$$\begin{aligned}&\partial_x((1-x)^{\alpha}(1+x)^{\beta}\partial_x v(x))+(1-x)^{\gamma}(1+x)^{\delta}v(x)=f(x),\\&\quad 0\leqslant\alpha\leqslant\gamma+2,0\leqslant\beta\leqslant\delta+2,|x|<1.\end{aligned}$$

Especially, they are more appropriate for singular problems. Moreover, we may use variable transformations to change some problems on unbounded and axisymmetrical domains to singular problems on bounded domains (see [5,14,15]), and then design suitable numerical algorithms and analyze numerical errors by using some results in this paper.

The results of this paper also improved the work of [17] essentially. Clearly, the power of $N(N+\alpha+\beta)$ in all approximation results is optimal. In particular, the $L^2_{\chi^{(\alpha,\beta)}}(\Lambda)$-errors of the orthogonal projection $P_{N,\alpha,\beta}$ and various interpolations depend now only on $||\partial_x^r v||_{\chi^{(\alpha+r,\beta+r)}}$. It is optimal and ensures approximability of some important functions related to singularity analysis of functions at corners. For instance, for a function $v(x)$ behaving like $(1-x)^\gamma$ as $x\to 1$, we have $\partial_x^r v(x) \sim (1-x)^{\gamma-r}$, and so for $r<2\gamma+1$, $||\partial_x^r v||_{\chi(r,0)}<\infty$. Therefore by the results in Sections 2 and 4, we have at least that for any arbitrary $\varepsilon>0$,

$$||P_{N,0,0}v-v||_{\chi(0,0)}, \quad ||\mathcal{I}_{Z,N,0,0}v-v||_{\chi(0,0)} \leqslant cN^{-2\gamma-1+\varepsilon}, \quad Z=G,R,L.$$

It is more important that the $H^1_{\alpha,\beta,\gamma,\delta}(\Lambda)$-errors of various orthogonal projections discussed in Section 3 only depend on the semi-norm $||\partial_x^r v||_{\chi^{(\alpha+r-1,\beta+r-1)}}$. Thus we can use the results in Sections 3 and 4 to analyze spectral methods and p-versions of finite element methods of differential equations of second order, and obtain optimal error estimates which also depend only on $||\partial_x^r v||_{\chi^{(\alpha+r-1,\beta+r-1)}}$. This semi-norm depends in turn on regularity of exact solution. In opposite, exact solution may not possess the regularity required by validity of approximation results in [17]. This fact also simplifies the analysis of various rational approximations induced by Jacobi polynomials, see [18,19].

In this paper, we described the explicit dependance of approximation results on the parameters α,β,γ and δ precisely. It helps us to deal with more complicated problems. For example, the convergence of orthogonal approximation on a triangle $\mathcal{T}=\{(x,y)\,|\,0\leqslant x,y\leqslant 1, 0\leqslant x+y\leqslant 1\}$, which is related to spectral methods and p-versions of finite element methods on non-rectangle domains, see [9,23]. In this case, we take the base functions

$$g_{l,m}(x)=2^{l+\frac{3}{2}}(1-y)^l J_l^{(0,0)}\left(\frac{2x+y-1}{1-y}\right)J_m^{(2l+1,0)}(2y-1).$$

Let $\mathcal{P}_{L,M}(\mathcal{T})=\mathrm{span}\{g_{l,m}(x,y)\,|\,0\leqslant l\leqslant L, 0\leqslant m\leqslant M\}$. The orthogonal projection $P_{L,M}: L^2(\mathcal{T})\to\mathcal{P}_{L,M}(\mathcal{T})$ is defined by

$$(P_{L,M}v-v,\phi)_{\mathcal{T}}=0 \quad \forall \phi\in\mathcal{P}_{L,M}(\mathcal{T}).$$

For $r,s\in\mathbb{N}$, we introduce the non-isotropic Sobolev space $H^{r,s}(\mathcal{T})$ with the norm,

$$||v||_{H^{r,s}(\mathcal{T})}=\left(\sum_{k=0}^{r}\sum_{j=0}^{k}||x^j y^{\frac{r}{2}}(1-y)^{k-j-\frac{r}{2}}\partial_x^j\partial_y^{k-j}v||^2_{L^2(\mathcal{T})} + ||x^{\frac{s}{2}}(1-x-y)^{\frac{s}{2}}\partial_x^s v||^2_{L^2(\mathcal{T})}\right)^{\frac{1}{2}}.$$

By using the results in Sections 2 and 4, we can derive some important approximation results, see [21]. For instance, if $M=O(L^{1+\frac{s}{2r}})$, then

$$||P_{L,M}v-v||_{\mathcal{T}}\leqslant cM^{-\frac{2rs}{2r+s}}||v||_{H^{r,s}(\mathcal{T})}.$$

But we cannot use the results in [17] for such problem. Indeed, all results in [17] are for fixed α, β, γ and δ, while one of parameters of $J_m^{(2l+1,0)}(2y-1)$ tends to infinity as $l \to \infty$.

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